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BOUNDS ON THE ORDER OF CROSS CHARACTERISTIC SUBGROUPS OF THE FINITE SIMPLE GROUPS OF LIE TYPE

TIHOMIR ASPAROUHOV

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Abstract. Let $X(r)$ and $G(q)$ be finite groups of Lie type and r and q be coprime. If $G(q)$ is embedded in $X(r)$, then the Landazuri-Seitz-Zalesskii theorem implies that $G(q)$ is small relative to $X(r)$. We formalize this observation and illustrate how it can be used with some applications.

1. Introduction

If X is a finite group of Lie type over a field of order r and G is a maximal subgroup of X, then it is essentially known that either G is a member of one of several nice families of subgroups of X or G is an almost simple group. If G is a finite simple group of Lie type over a field of order q with $(q, r) = 1$, then by the Landazuri-Seitz-Zalesskii theorem, the degree N of the minimal nontrivial representation of G over the nonnatural characteristic is very large compared to the degree of the minimal nontrivial representation of G over the natural characteristic. This fact has been used implicitly in various places in the literature to show that $|X : G|$ is very large. In this note we formalize this observation and make it explicit in Propositions 1 and 2. Then we illustrate the strength of the observation with four applications. For the most part these applications are not new. They are intended to give the reader a sense of how Propositions 1 and 2 could be used.

Notice that the lower bound on $|X : G|$ is large when N is large. On the other hand when N is small, one can hope to enumerate all cross characteristic subgroups of X.

Liebeck and Saxl, in [L] and [LS], have obtained bounds on the orders of the almost simple maximal subgroups of a finite simple group of Lie type, which enabled them to list all maximal subgroups of large order. They proved that when X is a classical group with natural projective module V of dimension n over $GF(r)$, then an almost simple maximal subgroup of X is of order less then r^{3n}, with some known exceptions. The bound $n^{12log_2 n}$, that we obtain in Proposition 2 for cross characteristic subgroups is stronger, in fact this bound is independent of r. The situation for the exceptional groups is similar.

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2. Main result

The following lemma is Lemma 2.1 in [TZ].

Lemma 1. *If* $2 \le r, 2 \le a_1 < a_2 < ... < a_k$ *are integers and* $\epsilon_1, \epsilon_2, ..., \epsilon_k \in \{1, -1\}$, *then*

$$\frac{1}{2} \le \frac{(r^{a_1}+\epsilon_1)(r^{a_2}+\epsilon_2)...(r^{a_k}+\epsilon_k)}{r^{a_1+a_2+...+a_k}} \le 2.$$

Let G *be a simple algebraic group over an algebraically closed field of characteristic* p. *Let* σ *be a surjective homomorphism* $G \to G$ *such that* G_σ *is finite. The finite groups* $O^{p'}(G_\sigma)$ *obtained this way are the finite groups of Lie type in characteristic* p.

Lemma 2. *If* $X(r)$ *is a finite simple group of Lie type, then*

$$r^{d-2} \le |X(r)| \le r^{d+1},$$

where d *is the dimension* $\overline{d}$ *of the corresponding algebraic simple group except for* ${}^2B_2(r), {}^2G_2(r)$ *and* ${}^2F_4(r)$, *where* $d = \overline{d}/2$ *(see the table,* r *is a power of a prime* p*). The lemma holds for* $PSp_4(2)', G_2(2)'$ *and* ${}^2F_4(2)'$ *too.*

$X(r)$	d	N
$PSL_n(r)$	n^2-1	n
$PSp_{2n}(r), n \ge 2$	$2n^2+n$	$2n$
$PSU_n(r), n \ge 3$	n^2-1	n
$PO^+_{2n}(r), n \ge 4$	$2n^2-n$	$2n$
$PO^-_{2n}(r), n \ge 4$	$2n^2-n$	$2n$
$PO_{2n+1}(r), n \ge 3, r-$odd	$2n^2+n$	$2n+1$
$E_6(r), {}^2E_6(r)$	78	27
$E_7(r)$	133	56
$E_8(r)$	248	248
$Sz(2^{2k+1})$	5	4
${}^3D_4(r)$	28	8
${}^2F_4(2^{2k+1})$	26	26
${}^2G_2(3^{2k+1})$	7	7
$F_4(r)$	52	$26-\delta_{p,3}$
$G_2(r)$	14	$7-\delta_{p,2}$

Proof. If $X(r) \ne {}^3D_4(q)$ we know that

$$|X(r)| = \frac{r^a}{s}(r^{a_1}+\epsilon_1)(r^{a_2}+\epsilon_2)...(r^{a_k}+\epsilon_k),$$

for some integers $1 \le a_1 < a_2 < ... < a_k$, such that $a + a_1 + ... + a_k = d$. The number s is the order of the center of the simply connected group of type $G(r)$. If $a_1 > 1$, then by Lemma 1 we get

$$\frac{r^d}{2s} \le |X(r)| \le \frac{2r^d}{s} \le r^{d+1}.$$

If $a_1 = 1$, then $G(r)$ is ${}^2B_2(r), {}^2G_2(r)$ or ${}^2F_4(r)$ and $r = r_1^l$ for $l > 1$. Now we can apply Lemma 1 for r_1 instead of r to get the same result. If $X(r) \ne {}^2E_6(2)$ and

$X(r) \neq PSU_n(2)$ we can easily check that $r^2 \geq 2s$ and this proves the lemma. If $X(r) = PSU_n(2)$, then $s \leq 3$ and therefore

$$|PSU_n(2)| \geq 2^{\frac{n(n-1)}{2}} \prod_{i=3}^{n}(2^i - (-1)^i) \geq 2^{n^2-3} = 2^{d-2},$$

because $(2^i+1)(2^{i+1}-1) \geq 2^{2i+1}$. If $X(r)$ is ${}^2E_6(2)$ or ${}^3D_4(q)$ the Lemma is easily verified.

Proposition 1. *Let $G(q)$ be a finite simple group of Lie type embedded in the group $PGL_n(K)$, where K is a field of characteristic coprime to q. Then*

$$|G(q)| \leq n^{10+2log_q n}.$$

Proof. This is a corollary of the Landazuri-Seitz-Zalesskii theorem [SZ] that gives a lower bound for n. Let $G(q)$ be $PSL_m(q)$ or $PSU_m(q)$. Then $n \geq q^{m-2}$ and therefore $m \leq log_q n + 2$. Also $d = m^2 - 1$ and by Lemma 2

$$|G| \leq q^{d+1} = q^{m^2} \leq q^{(log_q n)^2 + 2log_q n + 4} = n^{log_q n + 2} q^4 \leq n^{log_q n + 10}.$$

The last step is true since $m \geq 3$ implies $n \geq q$, and if $m = 2$, then $n \geq (q-1)/2$ and $q \leq 2n+1 \leq n^2$ except for $n = 2$ and $q = 5$, when the proposition is obvious. Let $G(q)$ be $PO_m(r)$ (m and r odd), $PSp_m(r)$ (m even) or $PO^{\pm}_m(r)$ (m even). Then $n \geq q^{m/2-1} \geq q$ and $d \leq (m^2+m)/2$. Now we get that $m \leq 2log_q n + 2$ and

$$|G| \leq q^{(m^2+m)/2+1} \leq q^{((2log_q n+2)^2 + 2log_q n + 2)/2+1} = n^{2log_q n+5} q^4 \leq n^{2log_q n + 9}.$$

Finally if $G(q)$ is an exceptional finite simple group of Lie type, then $n \geq q^{(d+1)/10}$. For example if $G(q) = E_8(q)$, then the LSZ theorem says that $n \geq q^{27}(q^2-1) \geq q^{26} \geq q^{24.9}$. Similarly for all the exceptional groups. We get that $|G(q)| \leq q^{d+1} \leq n^{10}$. Now we can easily check that the claim of the proposition holds for all the exceptions of the LSZ theorem and the proof is complete.

Proposition 2. *Let $G(q) \leq X(r)$ be two finite simple groups of Lie type and $(r,q) = 1$. Then*

$$|G(q)| \leq N^{12log_q N}$$

and

$$|X(r) : G(q)| \geq r^{(d-2-12log_r N \cdot log_q N)},$$

where d and N are from the table. In fact, N is the minimal degree of the representations of $X(r)$ over a field of the natural characteristic.

Proof. From the LSZ theorem if $G(q) \neq PSL_2(q)$, then $N \geq q$ and by Proposition 1

$$|G(q)| \leq N^{10+2log_q N} \leq N^{12log_q N}.$$

If $G = PSL_2(q)$, then $N^2 \geq q$ and $|G(q)| \leq N^6 \leq N^{12log_q N}$. The second part follows from the first part and Lemma 2.

3. Applications

A factorization of a finite group G is an expression $G = AB$ for some proper subgroups A and B of G. The factorization is maximal if both A and B are maximal. The maximal factorizations of the finite classical groups are found in [LPS]. Here we consider factorizations of a finite classical group by a group of Lie type in a different characteristic. We obtain restrictions for such factorizations.

Proposition 3. *Let $G(q)$ be a finite simple group of Lie type and $X_n(r)$ be one of the finite classical groups $PSL_n(r), PSU_n(r)$ $(n \geq 3)$, $PO_n(r)$ ($n \geq 7$ and r odd), $PSp_n(r)$ (n even ≥ 4), or $PO_n^{\pm}(r)$ (n even ≥ 8) with $(q, r) = 1$. If*

$$X_n(r) = H \cdot G(q)$$

is a factorization of $X_n(r)$, then $G(q)$ is one of the following groups: $PSL_2(q), q \leq 31; PSL_3(q), q \leq 4; PSL_4(2); PSp_4(q), q = 3, 5, 7; PSp_6(q), q = 2, 3; PSp_8(3); PSp_4(2)'; PSU_3(q), q = 3, 4; PSU_4(q), q = 2, 3; PSU_5(2); PO_8^+(2); PO_7(3); G_2(q), q = 3, 4; Sz(8)$. *In addition, r and n must satisfy one of the following: $r = 2$ and $n \leq 57$; $2 < r \leq 16$ and $r + n \leq 23$; $16 < r \leq 23$ and $n \leq 8$; $23 < r \leq 29$ and $n \leq 4$; or $29 < r \leq 59$ and $n = 2$.*

Proof. The smallest index of a subgroup of $X_n(r)$ is found in [Co] and is at least r^{n-2}. Therefore

$$n^{10+2log_q n} \geq |G(q)| \geq |X_n(r) : H| \geq r^{n-2}.$$

The above inequality is not true if $n \geq 137$ for any r and q, $(r, q) = 1$. Therefore $n \leq 136$. Let $l(G(q))$ be the minimal degree of the representations of $G(q)$ over a field of cross characteristic. Then the group $G(q)$ is restricted to the groups that satisfy the following condition: $l(G(q)) \leq 136$ and $|G(q)| \geq r^{n-2} \geq r^{l(G(q))-2}$ or if q is even then $|G(q)| \geq 3^{l(G(q))-2}$ and if q is odd then $|G(q)| \geq 2^{l(G(q))-2}$. The above list of groups consists of the finite simple groups of Lie type satisfying these two conditions. Now for a fixed r the possibilities for n are restricted by the following

$$l(G(q)) \leq n \leq log_r|G(q)| + 2 \tag{1}$$

for at least one of the groups $G(q), (q, r) = 1$ in the above list. Let $d(G(q))$ be the minimal degree of a nontrivial projective representation of $G(q)$ over the field of complex numbers. For the finite classical groups $d(G(q))$ can be found in [TZ]. If $(r, |G(q)|) = 1$, then additional restriction for n is $d(G(q)) \leq n$. Now if $r = 2$, the inequalities in (1) amount to $n \leq 57$. Similarly if $r = 3$ we get $n \leq 20$ and so on for $r = 4, ..., 16$. If $r > 16$ we get $n \leq 8$. If $r > 23$ we get $n \leq 6$. If $n \leq 6$ and $r > 11$, then by [Co] the smallest index of a subgroup of $X_n(r)$ is at least $(r^n - 1)/(r - 1)$ and therefore n is restricted to

$$(r^n - 1)/(r - 1) \leq |G(q)|. \tag{2}$$

Now by (1) and (2), if $r > 23$ then $n \leq 4$. Finally if $r > 29$ then n must be 2. There are two possibilities: $G = PSL_2(4)$ and $r + 1 \leq |G| = 60$, or $G = PSL_3(2)$, $r + 1 \leq |G| = 168$ and $(r, |G|) \neq 1$ (since $d(G) > 2$). These imply the final statement.

Lemma 3. *Let $G = X_n(r)$ be a finite classical group as in Proposition 3. If G acts transitively on a set Ω and the stabilizer of a point is isomorphic to a finite simple group of Lie type $H = G(q)$ for $(q, r) = 1$, then*

(1) for any element $1 \neq g \in G$, the fixed point ratio

$$\frac{f(g)}{|\Omega|} \leq \frac{n^{10+2log_q n}}{r^{n-2}},$$

where $f(g)$ is the number of elements of Ω fixed by g;
(2) the permutation rank of G on Ω satisfies

$$rank(G) \geq r^{n^2/4-24log_r nlog_q n}.$$

Proof. By Proposition 1, the fixed point ratio

$$\frac{f(g)}{|\Omega|} = \frac{|g^G \cap H|}{|g^G|} \leq \frac{|H|}{|G : C_G(g)|} \leq \frac{n^{10+2log_q n}}{r^{n-2}}.$$

Also $|\Omega| \leq \text{rank}(G)|H|$ and therefore

$$\text{rank}(G) \geq \frac{|G|}{|H|^2} \geq r^{d-2-24log_q nlog_r n} \geq r^{n^2/4-24log_r nlog_q n}.$$

Let G be a transitive permutation group on the set Ω, $|\Omega| = m$. For a permutation $g \in G$ let $\text{orb}(g)$ be the number of orbits of g on Ω and the index of g be $\text{Ind}(g) = m - \text{orb}(g)$. A genus l system is a triple (G, Ω, S), where $S = (g_i \neq 1 : i = 1, ..., r)$ is a generating set for G such that $g_1 g_2 ... g_r = 1$ and

$$2(m + l - 1) = \sum_{i=1}^{r} \text{Ind}(g_i).$$

This condition is equivalent to the existence of a branched covering of the Riemann sphere by a Riemann surface of genus l with monodromy group G. The study of genus 0 groups has been reduced in [A1], [GT] and [Sh] to the study of primitive almost simple groups G such that $f(g_i)/m > 1/85$ for some i.

Proposition 4. *If $n > 145$ the permutation group described in Lemma 3 is not a genus 0 group.*

Proof. If $n > 145$, then Lemma 3 (1) implies that for any $1 \neq g \in G$

$$\frac{f(g)}{m} < \frac{1}{85}.$$

Since $\text{orb}(g) = \sum_{i=1}^{d} f(g^i)/d$, where d is the order of g, we get that

$$\text{Ind}(g) \geq m - (d-1)m/(85d) - m/d = \frac{84}{85} m(1 - 1/d).$$

Now if G is a genus 0 group with some $S = (g_i : i = 1, ..., r; d_i = |g_i|)$, then

$$2m - 2 = \sum_{i=1}^{r} \text{Ind}(g_i) \geq \frac{84}{85} m(r - \sum_{i=1}^{r} 1/d_i).$$

This inequality forces r to be less than 5, and if $r = 4$ then $d_1 = ... = d_4 = 2$ and if $r = 3$ then $\sum 1/d_i \geq 1$. Now (9.6) in [GT] states that G is solvable or is isomorphic to A_5, which is impossible. □

4. Bounds on the Sylow subgroups

The following application of our main proposition was suggested by R. Guralnick. We want to formalize the observation that if a simple subgroup of a finite simple group of Lie type of characteristic p has a large p-subgroup, then it has to be a group of Lie type of the same characteristic. For a prime number p define $e_p(n)$ to be the maximal i such that p^i divides n and let $\Phi_k(x)$ be the k-th cyclotomic polynomial. We will need the following well known number theory result; see [M] page 27.

Lemma 4. *Let p be a prime number and q be a power of a different prime number. Then $p|\Phi_k(q)$ if and only if $k = a \cdot p^j$, where a is the minimal number i such that $p|(q^i-1)$. In this case $e_p(\Phi_a(q)) = e_p(q^a-1)$ and for $j > 0$, $e_p(\Phi_{ap^j}(q)) = 1$, except for $e_2(\Phi_{2a}(q)) = e_2(q+1)$ if $4|(q+1)$.*

Lemma 5. *Let j be $e_p(\prod_{i=1}^n(q^i-1))$. Then $p^j < q^{pn/(p-1)}$.*

Proof. Let a be as in Lemma 4. Then $p|q^a-1$ and therefore $p \le q^a$. Since

$$\prod_{i=1}^n (q^i-1) = \prod_{i=1}^n \Phi_i(q)^{[\frac{n}{i}]},$$

we can use Lemma 4 to estimate the p-part of the above product. If $p > 2$

$$p^j < (q^a)^{[n/a]} \cdot (q^a)^{[n/(ap)]} \cdot (q^a)^{[n/(ap^2)]} \dots \le q^{n(1+p^{-1}+p^{-2}+\dots)} < q^{pn/(p-1)}.$$

When $p = 2$ a similar calculation yields the result.

Lemma 6. *Let P be a cross characteristic Sylow subgroup of a finite simple classical group X. If $X = PSL_n(q)$, then $|P| < q^{2n}$; if $X = O_{2n+1}(q)$, $PO_{2n}^{\pm}(q)$, $PSp_{2n}(q)$, then $|P| < q^{4n}$; and if $X = PSU_n(q)$, then $|P| < (q+1)^{2n-1}$.*

Proof. If $X \neq PSU_n(q)$, then this Lemma follows directly from Lemma 4. If $X = PSU_n(q)$, then we need to apply Lemma 4 for $-q$ instead of q and repeat the proof of the Lemma using $p \le q^a \pm 1 \le (q+1)^a$.

Lemma 7. *Let $X(r), G(q)$ and N be as in Proposition 2. Let $r = p^l$, p a prime number. If P is a p-Sylow subgroup of $G(q)$, then $|P| < N^{10}$.*

Proof. If G is an exceptional group, then from the proof of Proposition 1 we get $|P| < |G| < N^{10}$. If $G = PSU_m(q)$, then by Lemma 6 $|P| < (q+1)^{2m-1}$ and as in the proof of Proposition 1, $m \le log_q N + 2$ and we get

$$|P| < (q+1)^{(2log_q N+3)} < (q^2)^{5log_q N} = N^{10}.$$

Similar arguments apply for the rest of the classical groups.

Proposition 5. *Let $X(r)$ and N be as in Proposition 2, $r = p^l$ and p be a prime number. Let S be a simple subgroup of X and let P be its p-Sylow subgroup. If $|P| \ge N^{10}$, then S is an alternating group or a group of Lie type of the same characteristic. If also $|P| \ge p^{(N+2)/(p-1)}$, then S is a group of Lie type of the same characteristic.*

Proof. It's easy to observe that S is not a sporadic group. If S is the alternating group A_m, then the minimal projective module is of dimension at least $m-2$. Finally

$$|P| < p^{m/(p-1)} \le p^{(N+2)/(p-1)}.$$

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California Institute of Technology, Pasadena, California 91125
E-mail address: `tihomir@cco.caltech.edu`