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MULTIPLIER THEOREMS FOR HERZ TYPE HARDY SPACES

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ABSTRACT. In this paper, the authors establish a multiplier theorem for Herz type Hardy spaces.

Let T_m be a multiplier operator defined in terms of Fourier transforms by

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi)$$

for suitable functions f. It is well-known that there is a multiplier theorem for $H^1(\mathbb{R}^n)$ (see [FS]): if $\alpha > n/2$ and

$$\int_{R<|\xi|<2R} |D^\beta m(\xi)|^2 d\xi \le CR^{n-2|\beta|},\ 0 < R < \infty, \tag{1}$$

for all $|\beta| \le \alpha$, then T_m can be extended to be a bounded operator on $H^1(\mathbb{R}^n)$. That is, m is a bounded multiplier of $H^1(\mathbb{R}^n)$.

Fix a function $\eta \in C_0^\infty(\mathbb{R}^n)$ with $0 \le \eta \le 1$, $\eta = 1$ on $1/2 \le |\xi| \le 2$ and $\operatorname{supp}\eta \subset \{1/4 \le |\xi| \le 4\}$. For $\delta > 0$, let us denote

$$m_\delta(\xi) = m(\delta\xi)\eta(\xi).$$

It is easy to check that (1) is equivalent to

$$\sup_\delta \|\widehat{m_\delta}\|_{K_2^{\alpha,2}(\mathbb{R}^n)} < \infty, \tag{2}$$

where $K_2^{\alpha,2}(\mathbb{R}^n)$ is a non-homogeneous Herz space (see [BS]). By using some embedding relations on Herz spaces, A. Baernstein II and E. T. Sawyer [BS] weakened (2) into

$$\sup_\delta \|\widehat{m_\delta}\|_{K_1^{\varepsilon,1}(\mathbb{R}^n)} < \infty, \tag{3}$$

where $0 < \varepsilon < \alpha - \frac{n}{2}$. In fact, this is just a special case of their theorem. In [BS], Baernstein and Sawyer showed that m is a bounded multiplier of $H^1(\mathbb{R}^n)$ under an even weaker condition than (3); see Theorem 3b in [BS, page 21].

By using the technique of Herz type Hardy spaces developed by the authors in [LY1]-[LY3] and [Y], in this paper, we shall first establish a multiplier theorem for the homogeneous Herz type Hardy space $H\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$ which is introduced by the authors of this paper in [LY1]. Then as simple consequences of this theorem, a multiplier theorem for the corresponding non-homogeneous version of the space

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$H\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$ and the special case mentioned above of the multiplier theorem of Baernstein and Sawyer for $H^1(\mathbb{R}^n)$ will be deduced.

Now, for the reader's convenience, let us recall the definition of the Herz spaces here. For $k \in \mathbb{Z}$, let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $A_k = B_k \setminus B_{k-1}$. We also denote by χ_k the characteristic function of the set A_k.

Definition 1. Let $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$.

(i) The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined in terms of

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \equiv \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p}$$

by letting

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) \equiv \{f \in L_{loc}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty\}.$$

(ii) The non-homogeneous Herz space $K_q^{\alpha,p}(\mathbb{R}^n)$ is defined in terms of

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} \equiv \left\{ \|f\chi_{B_0}\|_{L^q(\mathbb{R}^n)}^p + \sum_{k=1}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p}$$

by letting

$$K_q^{\alpha,p}(\mathbb{R}^n) \equiv \{f \in L_{loc}^q(\mathbb{R}^n) : \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} < \infty\}.$$

Here the usual modification was made when $p = \infty$.

In what follows, when $p = 1,\ 1 < q < \infty$ and $\alpha = n(1-1/q)$, we shall abbreviate $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$, respectively, as $\dot{K}_q(\mathbb{R}^n)$ and $A^q(\mathbb{R}^n)$. The latter is also said to be the Beurling algebras; see [CL] and [GR].

Definition 2. Let $1 < q < \infty$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, let Gf be the grand maximal function of f (see [FS] for its definition).

(i) The Hardy space $H\dot{K}_q(\mathbb{R}^n)$ associated with the Herz space $\dot{K}_q(\mathbb{R}^n)$ is defined by

$$H\dot{K}_q(\mathbb{R}^n) \equiv \{f \in \mathcal{S}'(\mathbb{R}^n) :\ Gf \in \dot{K}_q(\mathbb{R}^n)\}.$$

In this case, we also define $\|f\|_{H\dot{K}_q(\mathbb{R}^n)} \equiv \|Gf\|_{\dot{K}_q(\mathbb{R}^n)}$.

(ii) The Hardy space $HA^q(\mathbb{R}^n)$ associated with the Beurling algebra $A^q(\mathbb{R}^n)$ is defined by

$$HA^q(\mathbb{R}^n) \equiv \{f \in \mathcal{S}'(\mathbb{R}^n) :\ Gf \in A^q(\mathbb{R}^n)\}.$$

In this case, we also define $\|f\|_{HA^q(\mathbb{R}^n)} \equiv \|Gf\|_{A^q(\mathbb{R}^n)}$.

We remark that $HA^q(\mathbb{R}^n)$ was first introduced by Chen and Lau in [CL] for $n = 1$, and then by García-Cuerva in [GR] for $n > 1$. Obviously, $H\dot{K}_q(\mathbb{R}^n)$ is a homogeneous version of $HA^q(\mathbb{R}^n)$. Moreover, in [LY1], the authors proved that

$$HA^q(\mathbb{R}^n) = H\dot{K}_q(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \tag{4}$$

and

$$\|f\|_{HA^q(\mathbb{R}^n)} \sim \|f\|_{H\dot{K}_q(\mathbb{R}^n)} + \|f\|_{L^q(\mathbb{R}^n)}. \tag{5}$$

It is also well-known that $HA^q(\mathbb{R}^n) \subsetneqq H\dot{K}_q(\mathbb{R}^n) \subsetneqq H^1(\mathbb{R}^n)$ for any $q \in (1, \infty)$.

Let us now formulate our multiplier theorem for $H\dot{K}_q(\mathbb{R}^n)$.

Theorem 1. *Let $q \in (1, \infty)$ and m satisfy*

$$M \equiv \sup_{\delta} \|\widehat{m_\delta}\|_{K_1^{n(1-1/q),1}(\mathbb{R}^n)} < \infty. \tag{6}$$

Then m is a bounded multiplier of $H\dot{K}_q(\mathbb{R}^n)$.

By Corollary 2 in [BS, page 22], we know that if m satisfies the condition of Theorem 1, then m is a bounded multiplier of $L^q(\mathbb{R}^n)$ for $1 < q < \infty$. Therefore, from (4), (5) and Theorem 1, we have the following simple corollary.

Corollary 1. *Let $q \in (1, \infty)$ and m satisfy (6). Them m is a bounded multiplier of $HA^q(\mathbb{R}^n)$.*

The proof of Theorem 1 is based on the decomposition characterizations of Herz spaces and Herz type Hardy spaces in terms of central units and central atoms respectively. Let us recall that a function $e(x)$ is said to be a central (α, q) unit of restrict type if e satisfies

i) $\operatorname{supp} e \subset B(0, r), r \geq 1$;
ii) $\|e\|_{L^q(\mathbb{R}^n)} \leq |B(0, r)|^{-\alpha/n}$.

Lemma 1 ([LY2]). *Let $0 < \alpha < \infty$, $0 < p < \infty$ and $1 \leq q < \infty$. Then $f \in K_q^{\alpha,p}(\mathbb{R}^n)$ if and only if f can be expressed as*

$$f(x) = \sum_{k=0}^{\infty} \lambda_k e_k(x),$$

where each e_k is a central (α, q) unit of restrict type supported on B_k and $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$. Moreover,

$$\inf \Big\{ \Big(\sum_k |\lambda_k|^p \Big)^{1/p} \Big\} \sim \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)},$$

where the infimum is taken over all of the above decompositions of f.

Let us now turn to the definition of central atoms. A function $a(x)$ is said to be a central $(1, q)$ atom if a satisfies

(i) $\operatorname{supp} a \subset B(0, r), r > 0$;
(ii) $\|a\|_{L^q(\mathbb{R}^n)} \leq |B(0, r)|^{1/q-1}$;
(iii) $\int_{\mathbb{R}^n} a(x)dx = 0$.

Lemma 2 ([LY3]). *Let $1 < q < \infty$. Then $f \in H\dot{K}_q(\mathbb{R}^n)$ if and only if f can be expressed as*

$$f(x) = \sum_{k=-\infty}^{\infty} \lambda_k a_k(x),$$

where each a_k is a central $(1, q)$ atom supported on B_k and $\sum_{k=-\infty}^{\infty} |\lambda_k| < \infty$. Moreover,

$$\inf \Big\{ \sum_{k=-\infty}^{\infty} |\lambda_k| \Big\} \sim \|f\|_{H\dot{K}_q(\mathbb{R}^n)},$$

where the infimum is taken over all of the above decompositions of f.

To prove Theorem 1, we still need a lemma. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space of functions. In what follows, we let $\widehat{a_\delta}(\xi) \equiv \widehat{a}(\delta\xi)\psi(\xi)$.

Lemma 3. *Let a be a central $(1,q)$ atom supported on $B(0,1)$ and $b_j = (\widehat{a_{2^j}})^\vee$. Then for any given $d > 0$, we have the following three facts:*
(i) $\|b_j\|_{L^q(\mathbb{R}^n)} \le C2^{-nj(1-1/q)}$.
(ii) $|b_j(x)| \le C_d 2^{-nj(1-1/q)}|x|^{-d}$, for $|x| \ge 2^{j+1}$.
(iii) $|b_j(x)| \le C_d 2^j(1+|x|)^{-d}$, for all x and $j \le 0$.

Proof. Since $1 < q < \infty$ and

$$b_j(x) = 2^{-nj}\int_{|x-y|<2^j} a(2^{-j}(x-y))\widehat{\psi}(y)dy,$$

it follows from the generalized Minkowski inequality that

$$\begin{aligned}\|b_j\|_{L^q(\mathbb{R}^n)} &\le 2^{-nj}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}|a(2^{-j}(x-y))|^q dx\right\}^{1/q}|\widehat{\psi}(y)|dy\\ &\le C2^{-nj}2^{nj/q}\|a\|_{L^q(\mathbb{R}^n)}\|\widehat{\psi}\|_{L^1(\mathbb{R}^n)} \le C2^{-nj(1-1/q)}.\end{aligned}$$

Thus, (i) holds. Let us now assume $|x| \ge 2^{j+1}$. Note that $|y| \ge |x|/2$ and $\widehat{\psi} \in \mathcal{S}(\mathbb{R}^n)$. Then we have

$$\begin{aligned}|b_j(x)| &\le 2^{-nj}\int_{|x-y|<2^j}|a(2^{-j}(x-y))|\cdot|\widehat{\psi}(y)|dy\\ &\le 2^{-nj}\Big(\int_{|x-y|<2^j}|a(2^{-j}(x-y))|^q dy\Big)^{1/q}\Big(\int_{|x-y|<2^j}|\widehat{\psi}(y)|^{q'}dy\Big)^{1/q'}\\ &= 2^{nj(1/q-1)}\|a\|_{L^q(\mathbb{R}^n)}\Big(\int_{|y|\ge|x|/2}|\widehat{\psi}(y)|^{q'}dy\Big)^{1/q'}\\ &\le C_d 2^{nj(1/q-1)}|x|^{-d}.\end{aligned}$$

Thus, (ii) also holds. Finally, let us assume $j \le 0$. Since $\int_{\mathbb{R}^n} a(y)dy = 0$, we have

$$b_j(x) = \int_{|y|<1} a(y)\{\widehat{\psi}(x-2^j y) - \widehat{\psi}(x)\}dy.$$

It follows from the mean value theorem that there exists a $\theta \in (0,1)$ such that

$$|b_j(x)| \le \int_{|y|<1}|a(y)|\cdot|(\nabla_x\widehat{\psi})(x-\theta 2^j y)|2^j|y|dy,$$

where $\nabla_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$. Note that $1+|x-\theta 2^j y| \ge (1+|x|)/2$ and $\widehat{\psi} \in \mathcal{S}(\mathbb{R}^n)$. Then we have

$$|b_j(x)| \le C_d 2^j(1+|x|)^{-d}\int_{|y|<1}|y|\cdot|a(y)|dy \le C_d 2^j(1+|x|)^{-d}.$$

This completes the proof of the lemma. ∎

Proof of Theorem 1. By the decompositions of $H\dot{K}_q(\mathbb{R}^n)$ in terms of central atoms, it suffices to prove that the inequality

$$\|T_m a\|_{H\dot{K}_q(\mathbb{R}^n)} \le C \tag{7}$$

holds for all central $(1,q)$ atoms $a(x)$. Let $a(x)$ be a central $(1,q)$ atom. Since M is invariant for all dilations of m, we may assume $\operatorname{supp} a \subset B(0,1)$. In Lemma 3, if we take $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi \subset \{\xi : 1/2 \le |\xi| \le 2\}$, $0 \le \psi \le 1$, and $\sum_{j\in\mathbb{Z}} \psi(\xi 2^{-j}) = 1$, then $\psi\eta = \psi$ and

$$\begin{aligned} m(\xi)\widehat{a}(\xi) &= \sum_{j=-\infty}^{\infty} m(\xi)\eta(2^{-j}\xi)\widehat{a}(\xi)\psi(2^{-j}\xi) \\ &= \sum_{j=-\infty}^{\infty} m_{2^j}(2^{-j}\xi)\widehat{a_{2^j}}(2^{-j}\xi), \end{aligned}$$

where $\widehat{a_\delta}(\xi) = \widehat{a}(\delta\xi)\psi(\xi)$. By letting $N_j \equiv (m_{2^j})^\vee$, we have

$$T_m a(x) = \sum_{j=-\infty}^{\infty} 2^{nj}(N_j * b_j)(2^j x). \tag{8}$$

Without loss of generality, we may assume $M = 1$. Thus, the inequality $\|\widehat{m_\delta}\|_{L^1(\mathbb{R}^n)} \le 1$ holds for any $\delta > 0$. Therefore, it follows from the Hausdorff-Young inequality that $\|m\|_{L^\infty(\mathbb{R}^n)} \le 1$ and

$$\|N_j\|_{L^1(\mathbb{R}^n)} \le \|N_j\|_{K_1^{n(1-1/q),1}(\mathbb{R}^n)} \le 1.$$

Let us first prove

$$\|T_m a\|_{\dot{K}_q(\mathbb{R}^n)} \le C, \tag{9}$$

where C is independent of a and a is a central $(1,q)$ atom with $\operatorname{supp} a \subset B(0,1)$. We write

$$\begin{aligned} \|T_m a\|_{\dot{K}_q(\mathbb{R}^n)} &= \sum_{k=-\infty}^{\infty} 2^{kn(1-1/q)}\|(T_m a)\chi_k\|_{L^q(\mathbb{R}^n)} \\ &= \sum_{k=-\infty}^{2} \cdots + \sum_{k=3}^{\infty} \cdots \equiv I_1 + I_2. \end{aligned}$$

Since m is a bounded multiplier of $L^q(\mathbb{R}^n)$ by Corollary 2 in [BS, page 22], we have

$$I_1 \le C \sum_{k=-\infty}^{2} 2^{kn(1-1/q)}\|a\|_{L^q(\mathbb{R}^n)} \le C \sum_{k=-\infty}^{2} 2^{nk(1-1/q)} \le C.$$

On the other hand, by (8), we have

$$\begin{aligned} I_2 &= \sum_{k=3}^{\infty} 2^{kn(1-1/q)}\|(T_m a)\chi_k\|_{L^q(\mathbb{R}^n)} \\ &\le \sum_{k=3}^{\infty}\sum_{j=-\infty}^{\infty} 2^{nj}2^{kn(1-1/q)}\|(N_j * b_j)(2^j\cdot)\chi_k(\cdot)\|_{L^q(\mathbb{R}^n)} \\ &= \sum_{j=-\infty}^{\infty}\sum_{l=j+3}^{\infty} 2^{nl(1-1/q)}\|(N_j * b_j)(\cdot)\chi_l(\cdot)\|_{L^q(\mathbb{R}^n)} \\ &= \sum_{j=-\infty}^{0}\sum_{l=j+3}^{\infty} \cdots + \sum_{j=1}^{\infty}\sum_{l=j+3}^{\infty} \cdots \equiv I_{2,1} + I_{2,2}. \end{aligned}$$

Let us first estimate $I_{2,1}$. By Lemma 1, N_j can be expressed as

$$N_j(x) = \sum_{k=0}^{\infty} \lambda_k^j e_k^j(x),$$

where each e_k^j is a central $(n(1-1/q), 1)$ unit of restrict type supported on B_k and

$$\inf\Big(\sum_{k=0}^{\infty} |\lambda_k^j|\Big) \sim \|N_j\|_{K_1^{n(1-1/q),1}(\mathbb{R}^n)}.$$

Thus,

$$\begin{aligned}
I_{2,1} &= \sum_{j=-\infty}^{0} \sum_{l=j+3}^{\infty} 2^{nl(1-1/q)} \|(N_j * b_j)(\cdot)\chi_l(\cdot)\|_{L^q(\mathbb{R}^n)} \\
&\le \sum_{j=-\infty}^{0} \sum_{l=j+3}^{\infty} 2^{nl(1-1/q)} \sum_{k=0}^{\infty} |\lambda_k^j| \cdot \|(e_k^j * b_j)\chi_l\|_{L^q(\mathbb{R}^n)} \\
&= \sum_{j=-\infty}^{0} \sum_{l=j+3}^{\infty} 2^{nl(1-1/q)} \sum_{k=0}^{\max\{l-2,0\}} |\lambda_k^j| \cdot \|(e_k^j * b_j)\chi_l\|_{L^q(\mathbb{R}^n)} \\
&\quad + \sum_{j=-\infty}^{0} \sum_{l=j+3}^{\infty} 2^{nl(1-1/q)} \sum_{k=\max\{l-1,0\}}^{\infty} |\lambda_k^j| \cdot \|(e_k^j * b_j)\chi_l\|_{L^q(\mathbb{R}^n)} \\
&\equiv I_{2,1}^1 + I_{2,1}^2.
\end{aligned}$$

By (iii) in Lemma 3 with $d = n + \varepsilon,\ 0 < \varepsilon < 1$, we have

$$\begin{aligned}
\|(e_k^j * b_j)\chi_l\|_{L^q(\mathbb{R}^n)} &\le C2^j \|e_k^j\|_{L^1(\mathbb{R}^n)} \Big(\int_{A_l} |x|^{-dq} dx\Big)^{1/q} \\
&\le C2^j 2^{-l(n+\varepsilon)} 2^{ln/q}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
I_{2,1}^1 &\le C \sum_{j=-\infty}^{0} 2^j \sum_{l=j+3}^{\infty} 2^{-l\varepsilon} \sum_{k=0}^{\infty} |\lambda_k^j| \\
&\le C \sum_{j=-\infty}^{0} 2^j 2^{-j\varepsilon} \|N_j\|_{K_1^{n(1-1/q),1}(\mathbb{R}^n)} \\
&\le C \sum_{j=-\infty}^{0} 2^{j(1-\varepsilon)} \le C.
\end{aligned}$$

By (iii) in Lemma 3 again, we have

$$\begin{aligned}
\|(e_k^j * b_j)\chi_l\|_{L^q(\mathbb{R}^n)} &\le \|e_k^j\|_{L^1(\mathbb{R}^n)} \|b_j\|_{L^q(\mathbb{R}^n)} \le C2^j |B(0, 2^k)|^{-(1-1/q)} \\
&\le C2^j 2^{-kn(1-1/q)}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned} I_{2,1}^2 &\le C \sum_{j=-\infty}^{0} 2^j \sum_{l=j+3}^{\infty} 2^{nl(1-1/q)} \sum_{k=\max\{l-1,0\}}^{\infty} |\lambda_k^j| 2^{-kn(1-1/q)} \\ &\le C \sum_{j=-\infty}^{0} 2^j \sum_{k=0}^{\infty} |\lambda_k^j| 2^{-kn(1-1/q)} \sum_{l=-\infty}^{k+1} 2^{nl(1-1/q)} \\ &\le C \sum_{j=-\infty}^{0} 2^j \|N_j\|_{K_1^{n(1-1/q),1}(\mathbb{R}^n)} \le C. \end{aligned}$$

Hence, we obtain $I_{2,1} \le C$.

We now estimate $I_{2,2}$. Let $x \in A_l,\ l \ge j+3$. Then,

$$\begin{aligned} (N_j * b_j)(x) &= \int_{|y|\le 2^{l-1}} N_j(y) b_j(x-y)dy \\ &\quad + [(N_j \chi_{\widetilde{A}_l}) * b_j](x) + \int_{|y|>2^{l+2}} N_j(y) b_j(x-y)dy, \end{aligned}$$

where $\widetilde{A}_l = A_{l-1} \cup A_l \cup A_{l+1}$. Note that if $|y| \le 2^{l-2}$ and $x \in A_l,\ l \ge j+3$, then $|x-y| \ge 2^{j+1}$ and $|x-y| \ge |x|/2$. Thus, it follows from (ii) in Lemma 3 that

$$|b_j(x-y)| \le C_d 2^{-nj(1-1/q)} |x|^{-d} \le C_d 2^{-nj(1-1/q)} 2^{-ld}.$$

Also, note that if $|y| > 2^{l+3}$ and $x \in A_l,\ l \ge j+3$, then $|x-y| \ge 2^{j+1}$ and $|x-y| \ge |y|/2$. Also, it follows from (ii) in Lemma 3 that

$$|b_j(x-y)| \le C_d 2^{-nj(1-1/q)} |y|^{-d} \le C_d 2^{-nj(1-1/q)} 2^{-ld}.$$

Thus, when $x \in A_l,\ l \ge j+3$, we have

$$\begin{aligned} |(N_j * b_j)(x)| &\le C_d 2^{-nj(1-1/q)} 2^{-ld} \|N_j\|_{L^1(\mathbb{R}^n)} + |(N_j \chi_{\widetilde{A}_l}) * b_j(x)| \\ &\le C_d 2^{-nj(1-1/q)-ld} + |(N_j \chi_{\widetilde{A}_l}) * b_j(x)|. \end{aligned}$$

Applying these estimates and (i) in Lemma 3 with $d = n + \varepsilon$ to $I_{2,2}$, we obtain

$$\begin{aligned} I_{2,2} &= \sum_{j=1}^{\infty} \sum_{l=j+3}^{\infty} 2^{nl(1-1/q)} \|(N_j * b_j)(\cdot)\chi_l(\cdot)\|_{L^q(\mathbb{R}^n)} \\ &\le C \sum_{j=1}^{\infty} \sum_{l=j+3}^{\infty} 2^{nl(1-1/q)} 2^{-nj(1-1/q)} 2^{-ld} 2^{ln/q} \\ &\quad + \sum_{j=1}^{\infty} \sum_{l=j+3}^{\infty} 2^{ln(1-1/q)} \|(N_j \chi_{\widetilde{A}_l}) * b_j(\cdot)\chi_l(\cdot)\|_{L^q(\mathbb{R}^n)} \\ &\le C \sum_{j=1}^{\infty} 2^{-j(d-n/q)} + \sum_{j=1}^{\infty} \sum_{l=j+3}^{\infty} 2^{ln(1-1/q)} \|N_j \chi_{\widetilde{A}_l}\|_{L^1(\mathbb{R}^n)} \|b_j\|_{L^q(\mathbb{R}^n)} \\ &\le C + C \sum_{j=1}^{\infty} 2^{-nj(1-1/q)} \|N_j\|_{K_1^{n(1-1/q),1}(\mathbb{R}^n)} \le C + C \sum_{j=1}^{\infty} 2^{-nj(1-1/q)} \le C. \end{aligned}$$

Now, (9) follows from the above estimates on $I_1, I_{2,1}$ and $I_{2,2}$.

Actually, it is easy to prove that (9) is true for any central $(1, q)$ atom. That is, the inequality

$$\|T_m \widetilde{a}\|_{\dot{K}_q(\mathbb{R}^n)} \le C \tag{10}$$

holds for any central $(1, q)$ atom $\widetilde{a}$. In fact, let us assume that $\operatorname{supp} \widetilde{a} \subset B(0, r)$. Obviously, there exists a $k_0 \in \mathbb{Z}$ such that $2^{k_0} < r \le 2^{k_0+1}$. If I_1 and I_2 in the proof of (9) are now replaced by

$$\sum_{k=-\infty}^{k_0+2} 2^{kn(1-1/q)} \|(T_m \widetilde{a})\chi_k\|_{L^q(\mathbb{R}^n)}$$

and

$$\sum_{k=k_0+3}^{\infty} 2^{kn(1-1/q)} \|(T_m \widetilde{a})\chi_k\|_{L^q(\mathbb{R}^n)}$$

respectively, then (10) can be proved by a method similar to that of proving (9).

To prove (7), by the characterization of $H\dot{K}_q(\mathbb{R}^n)$ in terms of Riesz transforms (see [Y]), it suffices to show

$$\sum_{j=1}^{n} \|R_j(T_m a)\|_{\dot{K}_q(\mathbb{R}^n)} \le C, \tag{11}$$

where C is independent of a and R_j is the j-th Riesz transform. Since Riesz transforms are bounded on $H\dot{K}_q(\mathbb{R}^n)$ (see [Y]), we have

$$R_j a(x) = \sum_k \lambda_k^j a_k^j(x)$$

and

$$\sum_k |\lambda_k^j| \le C \|R_j a\|_{H\dot{K}_q(\mathbb{R}^n)} \le C,$$

where each a_k^j is a central $(1, q)$ atom and C is independent of a. Thus, it follows from (10) that

$$\begin{aligned}\sum_{j=1}^{n} \|R_j(T_m a)\|_{\dot{K}_q(\mathbb{R}^n)} &= \sum_{j=1}^{n} \Big\| \sum_k \lambda_k^j (T_m a_k^j) \Big\|_{\dot{K}_q(\mathbb{R}^n)} \\ &\le \sum_{j=1}^{n} \sum_k |\lambda_k^j| \cdot \|T_m a_k^j\|_{\dot{K}_q(\mathbb{R}^n)} \le C \sum_{j=1}^{n} \sum_k |\lambda_k^j| \le C.\end{aligned}$$

Thus, (11) holds. This completes the proof of Theorem 1. □

Let us now point out that if a linear operator T commutes with translations, then the boundedness of T on $H\dot{K}_q(\mathbb{R}^n)$ implies its boundedness on $H^1(\mathbb{R}^n)$. Precisely, we have

Theorem 2. *Let T be a linear operator that commutes with translations. If T is bounded on $H\dot{K}_q(\mathbb{R}^n)$, $1 < q < \infty$, then T is also bounded on $H^1(\mathbb{R}^n)$.*

Proof. By the atomic decomposition of $H^1(\mathbb{R}^n)$ (see [CW]), it suffices to prove that the inequality

$$\|Ta\|_{H^1(\mathbb{R}^n)} \le C$$

holds for any $(1,q)$ atom a. Let $a(x)$ be a $(1,q)$ atom supported on $B(x_0,r)$. That is, $a(x)$ satisfies the following conditions: $\operatorname{supp} a \subset B(x_0,r), \|a\|_{L^q(\mathbb{R}^n)} \le |B(x_0,r)|^{1/q-1}$, and $\int_{\mathbb{R}^n} a(x)dx = 0$. Let $\overline{a}(x) \equiv \tau_{-x_0}a(x) = a(x+x_0)$. It is easy to see that $\overline{a}$ is a central $(1,q)$ atom supported on $B(0,r)$. Thus, from the conditions of the theorem, it follows that

$$\|T\overline{a}\|_{H\dot{K}_q(\mathbb{R}^n)} \le C,$$

where C is independent of $\overline{a}$. Since T commutes with translations, we have

$$\begin{aligned}\|\tau_{-x_0}Ta\|_{H^1(\mathbb{R}^n)} &\le \|\tau_{-x_0}Ta\|_{H\dot{K}_q(\mathbb{R}^n)} = \|T\tau_{-x_0}a\|_{H\dot{K}_q(\mathbb{R}^n)}\\ &= \|T\overline{a}\|_{H\dot{K}_q(\mathbb{R}^n)} \le C.\end{aligned}$$

Thus, $\tau_{-x_0}Ta \in H^1(\mathbb{R}^n)$ and

$$\tau_{-x_0}Ta(x) = \sum_j \lambda_j a_j(x),$$

where each a_j is a $(1,q)$ atom, and $\sum_j |\lambda_j| \sim \|\tau_{-x_0}Ta\|_{H^1(\mathbb{R}^n)}$. Since $H^1(\mathbb{R}^n)$ is translation invariant, we then have $Ta \in H^1(\mathbb{R}^n)$ and

$$\|Ta\|_{H^1(\mathbb{R}^n)} \le \sum_j |\lambda_j| \le C.$$

This finishes the proof of Theorem 2. □

Note that if $q \in (1,\infty)$, then $\alpha = n(1-1/q) \in (0,n)$. As a simple corollary of Theorem 1 and Theorem 2, we have

Corollary 2. *Let $0 < \varepsilon < n$. If m satisfies*

$$\sup_\delta \|\widehat{m_\delta}\|_{K_1^{\varepsilon,1}(\mathbb{R}^n)} < \infty,$$

then m is a bounded multiplier of $H^1(\mathbb{R}^n)$.

Finally, we point out that it is still an open problem whether (6) is a necessary condition for an $L^\infty(\mathbb{R}^n)$ function m to be a bounded multiplier of $H\dot{K}_q(\mathbb{R}^n)$ in any sense (see [BS]). And we will discuss the similar problems of multipliers on general Herz type Hardy spaces $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ in a future paper.

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