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ON A CONJECTURE OF F. MÓRICZ

G. BROWN AND K. Y. WANG

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ABSTRACT. F. Móricz has investigated the integrability of double lacunary sine series. His result, valid for special lacunary sequences, does not extend in the form originally conjectured, but we establish a suitably modified result.

1. INTRODUCTION

Let $a_{ij}, i, j \in \mathbb{N}$, be real numbers satisfying the condition

$$\sigma = \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}^2 \right\}^{\frac{1}{2}} < \infty. \tag{1}$$

Suppose $q > 1$ and m_i, n_j are positive numbers satisfying

$$\frac{m_{i+1}}{m_i} \geq q, \quad \frac{n_{j+1}}{n_j} \geq q, \qquad m_1 = n_1 = 1,\ i, j \in \mathbb{N}. \tag{2}$$

Define

$$f(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \sin m_i x \sin n_j y,$$

$$g_j(x) = \sum_{i=1}^{\infty} a_{ij} \sin m_i x, \qquad h_j(y) = \sum_{j=1}^{\infty} a_{ij} \sin n_j y.$$

In the general case these limits are to be understood in the sense of L^2-convergence and, as Lemma 1 shows, there is no inherent ambiguity in the definition.

F. Móricz [3] considered the special case when $m_i = n_i = 2^{i-1}$, $i \in \mathbb{N}$. In this case he proved that the condition

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\sum_{k=i}^{\infty} \sum_{l=j}^{\infty} a_{kl}^2 \right)^{\frac{1}{2}} < \infty \tag{3}$$

is equivalent to

$$\frac{f(x, y)}{xy} \in L(0, 1)^2, \quad \frac{g_i(x)}{x} \in L(0, 1), \quad \frac{h_i(y)}{y} \in L(0, 1), \qquad i \in \mathbb{N}. \tag{4}$$

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He proposed that in the general case when m_i, n_j are positive integers satisfying condition (2) and the integrability condition, then (4) is satisfied if and only if

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\log\frac{m_{i+1}}{m_i}\log\frac{n_{j+1}}{n_j}\left(\sum_{k=i}^{\infty}\sum_{l=j}^{\infty}a_{kl}^2\right)^{\frac{1}{2}}<\infty. \tag{5}$$

Our result is the following

Theorem. *Let a_{ij}, m_i, n_j satisfy* (1) *and* (2)*. Let f, g_j, h_i be as above. Define*

$$S=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|, \qquad T=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\log\frac{m_{i+1}}{m_i}\left(\sum_{k=i+1}^{\infty}a_{kj}^2\right)^{\frac{1}{2}},$$

$$U=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\log\frac{n_{j+1}}{n_j}\left(\sum_{l=j+1}^{\infty}a_{il}^2\right)^{\frac{1}{2}},$$

$$V=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\log\frac{m_{i+1}}{m_i}\log\frac{n_{j+1}}{n_j}\left(\sum_{k=i+1}^{\infty}\sum_{l=j+1}^{\infty}a_{kl}^2\right)^{\frac{1}{2}}.$$

Then the condition (4) *is equivalent to the condition*

$$S+T+U+V<\infty. \tag{6}$$

We point out that in our theorem, m_i, m_j need not be integers. If $m_i = n_i = 2^{i-1}$, then (5) is equivalent to (6). But in general, as the following example shows, (5) is stronger than (6) and they are not equivalent.

First we note that the one-dimensional case is subsumed by the two-dimensional case. If we set $a_{ij} = b_i$ for $j = 1$ and $a_{ij} = 0$ for $j > 1$, then (5) reads

$$\sum_{i=1}^{\infty}\log\frac{m_{i+1}}{m_i}\left(\sum_{j=i}^{\infty}b_j^2\right)^{\frac{1}{2}}<\infty \tag{5$'$}$$

and (6) reads

$$\sum_{i=1}^{\infty}|b_i|+\sum_{i=1}^{\infty}\log\frac{m_{i+1}}{m_i}\left(\sum_{j=i+1}^{\infty}b_j^2\right)^{\frac{1}{2}}<\infty. \tag{6$'$}$$

Let $b_i = \{e^{-2^{i+1}} - e^{-2^{i+2}}\}^{1/2}$ and $m_i = \prod_{l=1}^{i} e^{e^{2^{l-1}}}$. Then we see that (6′) holds but (5′) does not.

We present the proof of the Theorem in two parts:

Theorem 1. *Let $d = 1 + \frac{4}{(q-1)^2} + \frac{2}{(q-1)}\sqrt{\frac{4}{(q-1)^2}+8}$. Then*

$$\int_0^d |g_j(x)|\frac{dx}{x}\le c_q\left\{\sum_{i=1}^{\infty}|a_{ij}|+\sum_{i=1}^{\infty}\log\frac{m_{i+1}}{m_i}\left(\sum_{k=i+1}^{\infty}a_{kj}^2\right)^{\frac{1}{2}}\right\}, \tag{7}$$

$$\int_0^d |h_i(y)| \frac{dy}{y} \le c_q \left\{ \sum_{j=1}^{\infty} |a_{ij}| + \sum_{j=1}^{\infty} \log \frac{n_{j+1}}{n_j} \left(\sum_{l=j+1}^{\infty} a_{il}^2 \right)^{\frac{1}{2}} \right\}, \tag{8}$$

$$\int_0^d \int_0^d |f(x,y)| \frac{dxdy}{xy} \le c_q(S+T+U+V). \tag{9}$$

This theorem demonstrates that (6) implies (4).

Theorem 2. (4) *implies* (6).

We point out that the number $d = d(q)$ in Theorem 1 is just the positive root of the equation: $\frac{1}{4}(d-1)^2 - \frac{2}{(q-1)^2}(d+1) - \frac{4}{(q-1)^2} = 0$. This particular definition of d will be retained throughout the paper.

2. Proof of Theorem 1

Lemma 1. *Let a, b be arbitrary real numbers and $Q = (a, a+d) \times (b, b+d)$. Then* $0.001\sigma \le \{\frac{1}{d^2} \int_Q f^2(x,y) dx\, dy\}^{\frac{1}{2}} \le \sigma$.

Proof. Let $I_{ijkl} = \int_Q \sin m_i x \sin m_k x \sin n_j y \sin n_l y \, dxdy$. We have $\int_Q f^2(x,y)\,dxdy = \sum a_{ij} a_{kl} I_{ijkl}$, where the sum $\sum$ is taken over $\mathbb{N}^4$. If $i \ne k$ and $j \ne l$, then

$$|I_{ijkl}| \le \left(\frac{1}{|m_i - m_k|} + \frac{1}{m_i + m_k} \right) \left(\frac{1}{|n_j - n_l|} + \frac{1}{n_j + n_l} \right).$$

Let S_1 denote the subset of $\mathbb{N}^4$ defined by $S_1 = \{(i,j,k,l) \in \mathbb{N}^4 \colon i \ne k, j \ne l\}$ and let $\sum_1$ denote the sum taken over S_1. By Schwarz's inequality $|\sum_1 s_{ij} a_{kl} I_{ijkl}| \le \{\sum_1 |a_{ij} a_{kl}|^2\}^{1/2} \{\sum_1 I_{ijkl}^2\}^{1/2}$. Applying condition (2) we find $\sum_1 I_{ijkl}^2 \le 16(\frac{1}{1-q^{-2}})^2(\frac{1}{q^2-1})^2$ and hence

$$|{\textstyle\sum_1} a_{ij} a_{kl} I_{ijkl}| \le \left\{ \frac{4}{(q-1)^2} - a_q \right\} \sigma^2,$$

where $a_q = \frac{4}{(q+1)(q-1)^2}$.

If $i = k$ and $j \ne l$, then $|I_{ijkl}| \le \frac{d+1}{2}(\frac{1}{|n_j - n_l|} + \frac{1}{n_j + n_l})$. Applying condition (2) again we find $\sum_2 |a_{ij} a_{kl} I_{ijkl}| \le \{\frac{2(d+1)}{(q-1)^2} - b_q\}\sigma^2$, where $\sum_2$ denotes the sum over the set $\{i \ne k, j = l\} \cup \{i = k, j \ne l\}$ and $b_q = \frac{2(d+1)}{(q-1)^2(q+1)}$.

Finally, if $(i,j) = (k,l)$, then by condition (2) we have

$$\begin{gathered} \frac{1}{4}\left\{ d - \max\left(|\sin d|, \frac{1}{q} \right) \right\}^2 \le I_{ijkl} \le \frac{1}{4}(d+1)^2, \\ \frac{1}{4}(d-1)^2 \sigma^2 \le \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sigma_{ij}^2 I_{ijij} \le \frac{1}{4}(d+1)^2 \sigma^2. \end{gathered} \tag{10}$$

Combining the estimates for $\sum_1, \sum_2$ and (10), noticing $\frac{1}{4}(d-1)^2 - \frac{2(d+1)}{(q-1)^2} - \frac{4}{(q-1)^2} = 0$ we get $\sqrt{a_q + b_q}\sigma \le \{\int_Q f^2(x,y)\,dx\,dy\}^{\frac{1}{2}} \le d\sigma$. If $1 < q \le 31$, then $\sqrt{a_q + b_q} > \frac{d}{850}$ and hence $\{\frac{1}{|Q|} \int_Q f^2(x,y)\,dx\,dy\}^{\frac{1}{2}} \ge \frac{\sigma}{850}$. If $q \ge 31$, then $d < 1.2$ and hence $I_{ijij} \ge \frac{1}{4}(d - \sin 1.2)^2$. So we modify (10) to get

$$\int_Q f^2(x,y)\,dx\,dy \ge \{a_q + b_q + 0.034^2\}\sigma^2$$

and hence for $q > 31$ we have $\{\frac{1}{|Q|}\int_Q f^2(x,y)\,dx\,dy\}^{\frac{1}{2}} > 0.028\sigma$. The combination of these estimates completes the proof. □

The following lemma is a direct corollary of Lemma 1.

Lemma 2. *Let a_j be real numbers satisfying $A = (\sum_{j=1}^{\infty} a_j^2)^{1/2} < \infty$ and let m_j be numbers satisfying the condition:*

$$m_1 = 1, \qquad \frac{m_{i+1}}{m_i} \geq q > 1. \tag{11}$$

Define $\psi(x) = \sum_{j=1}^{\infty} a_j \sin m_j x$. Then for any $a \in \mathbb{R}$

$$\frac{1}{200}A \leq \left\{\frac{1}{d}\int_a^{a+d} \psi^2(x)\,dx\right\}^{\frac{1}{2}} \leq 2A.$$

Proof of Theorem 1. We first prove (7). For fixed $j \in \mathbb{N}$, we omit the subscript j, and write $g = g_j$, $a_i = a_{ij}$ for simplicity. Then we have

$$\int_0^d x^{-1}|g(x)|\,dx \leq \sum_{i=1}^{\infty}\int_{\frac{d}{m_{i+1}}}^{\frac{d}{m_i}} x^{-1}\{|S_i(x)| + |T_i(x)|\}\,dx \tag{12}$$

where $S_i(x) = \sum_{k=1}^{i} a_k \sin m_k x,\ T_i(x) = \sum_{k=i+1}^{\infty} a_k \sin m_k x$. Since $|\sin m_k x| \leq m_k|x|$ we see $\int_{\frac{d}{m_{i+1}}}^{\frac{d}{m_i}} |S_i(x)|\frac{dx}{x} \leq d\sum_{k=1}^{i}|a_k|\frac{1}{q^{i-k}}$, and

$$\sum_{i=1}^{\infty}\int_{d/m_{i+1}}^{d/m_i} x^{-1}|S_i(x)|\,dx \leq c_q\sum_{i=1}^{\infty}|a_{ij}|. \tag{13}$$

Simultaneously, writing $c_k = a_{i+k,j}$, $u_k = \frac{m_{i+k}}{m_{i+1}}$ we get

$$\int_{\frac{d}{m_{i+1}}}^{\frac{d}{m_i}} |T_i(x)|\frac{dx}{x} = \int_d^{\frac{m_{i+1}}{m_i}d}\left|\sum_{k=1}^{\infty} c_k \sin u_k x\right|\frac{dx}{x}.$$

Let $m = [\frac{m_{i+1}}{m_i}]$ and $\psi(x) = \sum_{k=1}^{\infty} c_k \sin u_k x$. Then

$$\int_d^{\frac{m_{i+1}}{m_i}d} |\psi(x)|\frac{dx}{x} \leq \sum_{\mu=1}^{m}\int_{\mu d}^{(\mu+1)d}|\psi(x)|\frac{dx}{x}.$$

By Lemma 2 we have $\int_{\mu d}^{(\mu+1)d}|\psi(x)|\frac{dx}{x} \leq \frac{2}{\mu}(\sum_{k=i+1}^{\infty} a_{kj}^2)^{1/2}$ and hence

$$\int_{\frac{d}{m_{i+1}}}^{\frac{d}{m_i}} |T_i(x)|\frac{dx}{x} \leq c_q \log\frac{m_{i+1}}{m_i}\left\{\sum_{k=i+1}^{\infty} a_{kj}^2\right\}^{1/2}, \tag{14}$$

$$\sum_{i=1}^{\infty}\int_{d/m_{i+1}}^{d/m_i} x^{-1}|T_i(x)|\,dx \leq c_q\sum_{i=1}^{\infty}\log\frac{m_{i+1}}{m_i}\left(\sum_{k=i+1}^{\infty} a_{kj}^2\right)^{1/2}. \tag{15}$$

Combining (12), (13) and (15) we get (7). By symmetry (8) follows.

Now we write $I_{ij} = \int_{\frac{d}{m_{i+1}}}^{\frac{d}{m_i}} \int_{\frac{d}{n_{j+1}}}^{\frac{d}{n_j}} |f(x,y)| \frac{dxdy}{xy}$ and define

$$\phi_{kl}(x,y) = a_{kl} \sin m_k x \sin n_l y,$$

$$f_1(x,y) = \sum_{k=1}^{i} \sum_{l=1}^{j} \phi_{kl}(x,y), \qquad f_2(x,y) = \sum_{k=i+1}^{\infty} \sum_{l=1}^{j} \phi_{kl}(x,y),$$

$$f_3(x,y) = \sum_{k=1}^{i} \sum_{l=j+1}^{\infty} \phi_{kl}(x,y), \qquad f_4(x,y) = \sum_{k=i+1}^{\infty} \sum_{l=j+1}^{\infty} \phi_{kl}(x,y).$$

Then define $I_{ij}^{(\nu)} = \int_{d/m_{i+1}}^{d/m_i} \int_{d/n_{j+1}}^{d/n_j} |f_\nu(x,y)| \frac{dxdy}{xy}$, $\nu = 1,2,3,4$. We see $I_{ij}^{(1)} \leq d^2 \sum_{k=1}^{i} \sum_{l=1}^{j} |a_{kl}| \frac{1}{q^{i-k}} \cdot \frac{1}{q^{j-l}}$ and hence

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} I_{ij}^{(1)} \leq c_q \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| = c_q S. \tag{16}$$

Secondly, $I_{ij}^{(2)} \leq \sum_{l=1}^{j} \frac{d}{q^{j-l}} \int_{\frac{d}{m_{i+1}}}^{\frac{d}{m_i}} |\sum_{k=i+1}^{\infty} a_{kl} \sin m_k x| \frac{dx}{x}$. So by (14)

$$\int_{\frac{d}{m_{i+1}}}^{\frac{d}{m_i}} \left| \sum_{k=i+1}^{\infty} a_{kl} \sin m_k x \right| \frac{dx}{x} \leq c_q \log \frac{m_{i+1}}{m_i} \left(\sum_{k=i+1}^{\infty} a_{kl}^2 \right)^{\frac{1}{2}}.$$

Therefore

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} I_{ij}^{(2)} \leq c_q T. \tag{17}$$

Symmetrically we have

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} I_{ij}^{(3)} \leq c_q U. \tag{18}$$

Finally, to estimate $I_{ij}^{(4)}$ we change the integrating variables by writing $x = s/m_{i+1}$, $y = t/n_{j+1}$. Then we get

$$I_{ij}^{(4)} = \int_d^{(m_{i+1}/m_i)d} \int_d^{(n_{j+1}/n_j)d} \left| \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{kl} \sin u_k s \sin v_l t \right| \frac{dsdt}{st}$$

where $b_{kl} = a_{i+k,j+l}$, $u_k = \frac{m_{i+k}}{m_{i+1}}$, $v_l = \frac{n_{j+l}}{n_{j+1}}$. Let $m = [\frac{m_{i+1}}{m_i}]$ and $n = [\frac{n_{j+1}}{n_j}]$. Define

$$\theta_{\mu\nu} = \int_{\mu d}^{(1+\mu)d} \int_{\nu d}^{(1+\nu)d} \left| \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{kl} \sin u_k s \sin v_l t \right| \frac{dsdt}{st}.$$

Applying Lemma 1 we get $\theta_{\mu\nu} \leq \frac{1}{\mu\nu} (\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{kl}^2)^{1/2}$, and hence

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} I_{ij}^{(4)} \leq c_q V. \tag{19}$$

A combination of (16)–(19) yields (9). The proof is complete. □

3. Proof of Theorem 2

To prove Theorem 2 we need more integral estimates.

Lemma 3. *Let $m_i, c_{ij}, i, j \in \mathbb{N}$, be real numbers satisfying condition* (11) *and $B = (\sum_{i=1}^{\infty} c_{ii})^2 + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij}^2 < \infty$. Define*

$$h(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij}\cos(m_i - m_j)x.$$

Then, for any $a \in \mathbb{R}$, $\int_a^{a+d} h^2(x)\,dx \le c_q B$, where, and throughout this paper, c_q denotes a constant depending only on q.

Proof. Define

$$h_1(x) = \sum_{i=2}^{\infty}\sum_{j=1}^{i-1} c_{ij}\cos(m_i - m_j)x, \qquad h_2(x) = \sum_{j=2}^{\infty}\sum_{i=1}^{j-1} c_{ij}\cos(m_i - m_j)x.$$

We see $h^2(x) \le 3\{h_1^2(x) + h_2^2(x) + (\sum_{i=1}^{\infty} c_{ii})^2\}$. We have $h_1^2(x) = \frac{1}{2}\{g_1(x) + g_2(x)\}$ where

$$g_1(x) = \sum_{i>j}\sum_{k>l} c_{ij}c_{kl}\cos(m_i - m_j - m_k + m_l)x,$$
$$g_2(x) = \sum_{i>j}\sum_{k>l} c_{ij}c_{kl}\cos(m_i - m_j + m_k - m_l)x.$$

By condition (11) for $i > j$, $k > l$, we have $m_i - m_j + m_k - m_l \ge 2(q-1)\cdot\sqrt{m_{i-1}\cdot m_{k-1}}$. Hence by Schwarz's inequality we get

$$\int_d^{a+d} g_2(x)\,dx \le c_q \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij}^2. \tag{20}$$

Choose $n \in \mathbb{N}$ such that $1 - \frac{1}{q} - \frac{1}{q^n} = \delta > 0$. Then define subsets J_μ of $(i,j,k,l) \in \mathbb{N}^4$ by

$$J_1 = \{i > j, i \ge k+n, k > l\}, \qquad J_2 = \{i > j, i \le k-n, k > l\},$$
$$J_3 = \{k+n > i > k > l, i \ge j+n\},$$
$$J_4 = \{k+n > i > k > l, i > j > i-n\},$$
$$J_5 = \{i+n > k > i > j, k > l\}, \qquad J_6 = \{i = k > j, k > l\}$$

and define $g_{1\nu}(x) = \sum_\nu c_{ij}c_{kl}\cos(m_i - m_j - m_k + m_l)x$, $\nu = 1, \ldots, 6$, where the sum $\sum_\nu$ denotes $\sum_{(i,j,k,l)\in J_\nu}$. Then we see $g_1 = \sum_{\nu=1}^{6} g_{1\nu}$. First we have $\int_a^{a+d} g_{1l}(x)\,dx \le \sum_1 |c_{ij}c_{kl}|(1 - \frac{1}{q} - \frac{1}{q^n})^{-1}\frac{2}{m_i}$. So by Schwarz's inequality

$$\int_a^{a+d} g_{11}(x)\,dx \le c_q \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij}^2. \tag{21}$$

Symmetrically

$$\int_a^{a+d} g_{12}(x)\,dx \le c_q \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij}^2. \tag{22}$$

If $(i, j, k, l) \in J_3$, then $m_i - m_j - m_k + m_l \geq \delta m_i$. Hence we have

$$\int_a^{a+d} g_{13}(x)\,dx \leq \frac{2}{\delta} \sum_{k=2}^{\infty} \sum_{l=1}^{k-1} |c_{kl}| \sum_{i=\max(n+1,k+1)}^{\infty} \sum_{j=1}^{i-n} \frac{|c_{ij}|}{m_i}.$$

The using Schwarz's inequality we get

$$\int_a^{a+d} g_{13}(x)\,dx \leq c_q B. \tag{23}$$

For $(i, j, k, l) \in J_4$ let $\varepsilon_{ijkl} = |\int_a^{a+d} \cos(m_i - m_j - m_k + m_l)x\,dx|$. Then

$$\int_a^{a+d} g_{14}(x)\,dx \leq \sum_{k=2}^{\infty} \sum_{l=1}^{k-1} \sum_{i=k+1}^{k+n} \sum_{j=\max(1,i-n)}^{i-1} |c_{ij} c_{kl}| \varepsilon_{ijkl}.$$

If we define $\varepsilon_{ijkl} = c_{ij} = 0$ when $i \leq 0$ or $j \leq 0$, then we get

$$\int_a^{a+d} g_{14}(x)\,dx \leq \sum_{s=1}^{n} \sum_{t=0}^{n-1} \sum_{k=2}^{\infty} |c_{k+s,k+s-n+t}| \sum_{l=1}^{k-1} |c_{kl}| \eta_l(k, s, t),$$

where $\eta_l(k, s, t) = \varepsilon_{k+s,k+s-n+t,k,l}$ will be written as η_l for simplicity. Write $\sigma_k = \sum_{l=1}^{k-1} |c_{k,l}| \eta_l$. Then $\sigma_k \leq \{\sum_{l=1}^{k-1} c_{kl}^2 \sum_{l=1}^{k-1} \eta_l^2\}^{1/2}$. Define

$$l_0 = \min \left\{ l \in \{1, \ldots, k-1\} \colon |m_{k+s} - m_{k+s-n+1} - m_k + m_l| < \frac{1}{2} \left(1 - \frac{1}{q}\right) m_l \right\},$$

$l_0 = 0$ for the case that the minimum does not exist. As a convention we let $m_0 = 0$. Then for all $l \neq l_0$, $|m_{k+s} - m_{k+s-n+t} - m_k + m_l| \geq \frac{1}{2}(1 - \frac{1}{q}) m_l$. Hence we have $n_l \leq c_q \frac{1}{m_l}$, for $l \neq l_0$, $1 \leq l \leq k-1$, and $\eta_{l_0} \leq d$. Consequently we get $\{\sum_{l=1}^{k-1} \eta_l^2\}^{1/2} \leq c_q$, and $\sigma_k \leq c_q \{\sum_{l=1}^{k-1} c_{kl}^2\}^{1/2}$. Now we derive

$$\int_a^{a+d} g_{14}(x)\,dx \leq c_q B. \tag{24}$$

Since J_5 is symmetrically related to $J_3 \cup J_4$ we conclude

$$\int_a^{a+d} g_{15}(x)\,dx \leq c_q B. \tag{25}$$

By a direct calculation we get

$$\int_a^{a+d} g_{16}(x)\,dx \leq c_q B. \tag{26}$$

A combination of (21)–(26) yields

$$\int_a^{a+d} g_1(x)\,dx \leq c_q B. \tag{27}$$

Then we combine (20) and (27) to get $\int_a^{a+d} h_1^2(x)\,dx \leq c_q B$. Symmetrically we conclude $\int_a^{a+d} h_2^2(x)\,dx \leq c_q B$. And at last we derive $\int_a^{a+d} h^2(x)\,dx \leq c_q B$ as required. □

Lemma 4. *Under the assumptions of Lemma 3 define*

$$g(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij}\cos(m_i + m_j)x.$$

Then $\int_a^{a+d} g^2(x)\,dx \le c_q B$.

The proof is completely similar to that of Lemma 3. We omit it.

Lemma 5. *Under the conditions of Lemma 1*

$$\left\{\int_Q f^4(x,y)\,dxdy\right\} \le c_q\sigma.$$

Proof. For $i \in \mathbb{N}$ and $y \in \mathbb{R}$ fixed we define $b_i = \sum_{j=1}^{\infty} a_{ij}\sin n_j y$. Then we see $f^2(x,y) = \frac{1}{2}\{h(x) + g(x)\}$ where h, g are defined respectively as in Lemma 3 and Lemma 4 with coefficients $c_{ij} = b_i b_j$. Then by these lemmas we conclude

$$\int_Q f^4(x,y)\,dxdy \le c_q \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_b^{b+d}\{b_ib_j\}^2\,dy. \tag{28}$$

Since $b_ib_j = \frac{1}{2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{ij}a_{jl}\{\cos(n_k - n_l)y - \cos(n_k + n_l)y\}$ we apply Lemmas 3 and 4 again to get $\int_b^{b+d} b_i^2 b_j^2\,dy \le c_q\sum_{k=1}^{\infty} a_{ik}^2 \sum_{i=1}^{\infty} a_{jl}^2$. Substituting this into (28) we complete the proof. □

The following two estimates follow from Lemma 1 and Lemma 5.

Lemma 6. *Under the conditions of Lemma 1* $c_q\int_Q |f(x,y)|\,dxdy \ge \sigma$.

Lemma 7. *Under the assumptions of Lemma 2* $c_q\int_a^{a+d}|\psi(x)|\,dx \ge A$.

The following lemma is the essence of Móricz's Lemmas 2 and 3 of [3].

Lemma 8. *Let* $a_i \ge 0$, $i \in \mathbb{N}$. *Then for any* $r \in \mathbb{N}$ *and* $n > r$

$$\sum_{i=r+1}^{n} a_i \le \frac{1}{\sqrt{r}}\sum_{i=1}^{n-1}\left(\sum_{j=i+1}^{n} a_j^2\right)^{1/2}. \tag{29}$$

Proof of Theorem 2. By Lemma 1, (4) is equivalent to

$$\frac{f(x,y)}{xy} \in L(0,\lambda d)^2, \quad \frac{g_i(x)}{x} \in L(0,\lambda d), \quad \frac{h_i(y)}{y} \in L(0,\lambda d), \qquad i \in \mathbb{N}, \tag{4'}$$

where $\lambda = \max(1, \frac{1}{q-1}) \ge 1$ and d is the value defined in Theorem 1. Now assume (4′) holds; we are going to prove (6).

Let $A_{ij} = \log\frac{m_{i+1}}{m_i}(\sum_{k=i+1}^{\infty} u_{kj}^2)^{\frac{1}{2}}$, $B_{ij} = \log\frac{n_{j+1}}{n_j}(\sum_{l=j+1}^{\infty} a_{il}^2)^{\frac{1}{2}}$. We first prove $\sum_{k=1}^{\infty} A_{kj} < \infty$, $\sum_{l=1}^{\infty} B_{il} < \infty$ $(i, j \in \mathbb{N})$.

For n big enough and j fixed, let $I_n = \int_{\lambda d/m_{n+1}}^{\lambda d} |g_j(x)|\frac{dx}{x}$. Then $I_n \ge J_n - R_n$ where

$$R_n = \sum_{i=1}^{n}\int_{(\lambda d/m_{i+1})}^{(\lambda d/m_i)}\left|\sum_{k=1}^{i} a_{kj}\sin m_k x\right|\frac{dx}{x} \le c_q\sum_{i=1}^{n}|a_{ij}|$$

and

$$J_n = \sum_{i=1}^{n} \int_{\lambda d/m_{i+1}}^{\lambda d/m_i} \left| \sum_{k=i+1}^{\infty} a_{kj} \sin m_k x \right| \frac{dx}{x}.$$

Applying Lemma 7 we get

$$J_n \geq \sum_{i=1}^{n} \sum_{\mu=1}^{m} \frac{1}{(\mu+\lambda)} c_q \left(\sum_{k=i+1}^{\infty} a_{kj}^2 \right)^{\frac{1}{2}}$$

where $m = [(\frac{m_{i+1}}{m_i} - 1)\lambda] \in \mathbb{N}$. It is now clear why we take λd instead of d. We obtain, in fact, $J_n \geq c_q \sum_{i=1}^{n} A_{ij}$ and hence $\sum_{i=1}^{n} A_{ij} \leq c_q' \sum_{i=1}^{n} |a_{ij}| + c_q'' I_n$. If $\sum_{i=1}^{\infty} A_{ij} = \infty$, noticing $\int_0^{\lambda d} |g_j(x)| \frac{dx}{x} < \infty$ we derive

$$1 \leq \overline{\lim_{n\to\infty}} c_q' \left\{ \sum_{i=1}^{n} |a_{ij}| \left(\sum_{i=1}^{n} A_{ij} \right)^{-1} \right\}.$$

But by Lemma 8 we conclude the right part of this inequality should be zero. This contradiction shows $\sum_{i=1}^{\infty} A_{ij} < \infty$. Symmetrically we know $\sum_{j=1}^{\infty} B_{ij} < \infty$.

Next we define f_ν, $\nu = 1, 2, 3, 4$, as in the proof of Theorem 1 and define

$$E_{ij}^{(\nu)} = \int_{(\lambda d/m_{i+1})}^{(\lambda d/m_i)} \int_{(\lambda d/n_{j+1})}^{(\lambda d/n_j)} |f_\nu(x,y)| \frac{dxdy}{xy}, \qquad \nu = 1, 2, 3, 4,\ i, j \in \mathbb{N}.$$

For big $s, t \in \mathbb{N}$ let

$$\sigma_{s,t}^{(\nu)} = \sum_{i=1}^{s} \sum_{j=1}^{t} E_{ij}^{(\nu)}, \qquad \sigma_{s,t} = \sum_{i=1}^{s} \sum_{j=1}^{t} \int_{\lambda d/m_{i+1}}^{\lambda d/m_i} \int_{\lambda d/n_{j+1}}^{\lambda d/n_j} \frac{|f(x,y)|}{xy} \, dxdy.$$

We have $\sigma_{s,t} \geq \sigma_{st}^{(4)} - (\sigma_{st}^{(1)} + \sigma_{st}^{(2)} + \sigma_{st}^{(3)})$. By an argument similar to that used in the proof of Theorem 1 we get inequalities similar to (16)–(18), viz.

$$\sigma_{st}^{(1)} \leq c_q \sum_{i=1}^{s} \sum_{j=1}^{t} |a_{ij}|, \qquad \sigma_{st}^{(2)} \leq c_q \sum_{i=1}^{s} \sum_{j=1}^{t} \log \frac{m_{i+1}}{m_i} \left(\sum_{k=j+1}^{\infty} a_{kj}^2 \right)^{\frac{1}{2}},$$

$$\sigma_{st}^{(3)} \leq c_q \sum_{i=1}^{s} \sum_{j=1}^{t} \log \frac{n_{j+1}}{n_j} \left(\sum_{l=j+1}^{\infty} a_{il}^2 \right)^{\frac{1}{2}}.$$

We now estimate $\sigma_{st}^{(4)}$ applying Lemma 6. Let $m = [(\frac{m_{i+1}}{m_i} - 1)\lambda]$ and $n = [(\frac{n_{j+1}}{n_j} - 1)\lambda]$. Then $m \in \mathbb{N}, n \in \mathbb{N}$ and

$$\sigma_{st}^{(4)} \geq \sum_{i=1}^{s} \sum_{j=1}^{t} \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} \int_{\lambda d+(\mu-1)d}^{\lambda d+\mu d} \int_{\lambda d+(\nu-1)d}^{\lambda d+\nu d} S_{ij}(x,y) \, dxdy$$

where

$$S_{ij}(x,y) = x^{-1} y^{-1} \left| \sum_{k=i+1}^{\infty} \sum_{l=j+1}^{\infty} a_{kl} \sin \frac{m_k}{m_{i+1}} x \sin \frac{n_l}{n_{j+1}} y \right|.$$

Hence by Lemma 6 we get

$$\sigma_{st}^{(4)} \geq c_q \sum_{i=1}^{s} \sum_{j=1}^{t} \log \frac{m_{i+1}}{m_i} \log \frac{n_{j+1}}{n_j} \left(\sum_{k=i+1}^{\infty} \sum_{l=j+1}^{\infty} a_{kl}^2 \right)^{\frac{1}{2}}. \tag{30}$$

Then applying Lemma 8, by an argument similar to that for the proof of $\sum_{i=1}^{\infty} A_{ij} < \infty$ we conclude $V < \infty$. Then noticing

$$\begin{aligned}
T &= \sum_{i=1}^{\infty} A_{i1} + \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} A_{ij} \leq \sum_{i=1}^{\infty} A_{i1} + c_q V, \\
U &= \sum_{j=1}^{\infty} B_{1j} + \sum_{j=1}^{\infty} \sum_{i=2}^{\infty} B_{ij} \leq \sum_{j=1}^{\infty} B_{1j} + c_q V, \\
S &= \sum_{j=2}^{\infty} |a_{ij}| + \sum_{i=2}^{\infty} |a_{i1}| + |a_{il}| + \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} |a_{ij}| \\
&\leq c_q \sum_{j=1}^{\infty} B_{1j} + c_q \sum_{i=1}^{\infty} A_{i1} + |a_{11}| + c_q V,
\end{aligned}$$

we derive (6). The proof is complete. □

Remarks. (a) In our argument the series by which we define functions need not be trigonometric series because the coefficients m_i, n_j need not be integers. We must understand such series in the sense of L^2-convergence. For example, if conditions (1) and (2) are satisfied, then the "partial sums"

$$S_{\mu\nu}(x,y) = \sum_{i=1}^{\mu} \sum_{j=1}^{\nu} a_{ij} \sin m_i x \sin n_j y$$

converge in $L^2(Q)$ for any compact set $Q \subset \mathbb{R}^2$. This is a consequence of Lemma 1. Meanwhile we can easily demonstrate that the convergence of $S_{\mu\nu}$ in $L^2(Q)$ does not depend on the manner in which μ and ν tend to infinity. For a discussion of different kinds of multiple limits we refer the reader to [4].

(b) Since $S_{\mu\nu}$ can be non-trigonometric sums it does not appear to be a trivial question whether the convergence of $S_{\mu\nu}$ in L^2 implies almost everywhere convergence.

(c) Our result can be extended to higher dimensional cases in a quite straightforward manner.

(d) Since this paper was submitted in June, 1993, two related papers of interest have appeared, viz. [1], [2]. We thank the referee for providing the details.

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Vice-Chancellor, University of Sydney, Sydney, New South Wales 2006, Australia
E-mail address: m.jackson@vcc.usyd.edu.au

Department of Mathematics, Beijing Normal University, 100871 Beijing, People's Republic of China
E-mail address: wangky@email.bnu.edu.cn