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ZEROS OF THE ZAK TRANSFORM ON LOCALLY COMPACT ABELIAN GROUPS

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ABSTRACT. Let G be a locally compact abelian group. The notion of Zak transform on $L^2(\mathbb{R}^d)$ extends to $L^2(G)$. Suppose that G is compactly generated and its connected component of the identity is non-compact. Generalizing a classical result for $L^2(\mathbb{R})$, we then prove that if $f \in L^2(G)$ is such that its Zak transform Zf is continuous on $G \times \widehat{G}$, then Zf has a zero.

1. INTRODUCTION

The Zak transform on the real line, sometimes also referred to as the Weil-Brezin map, was introduced in 1967 by Zak [11] to construct a quantum mechanical representation for the description of the motion of a Bloch electron in the presence of a magnetic or electric field. Subsequently it proved to be an important tool in applied areas such as signal theory, wavelet analysis and solid state physics (compare the survey article [7] and the references therein).

For $f \in L^2(\mathbb{R})$ the Zak transform Zf is the function on $\mathbb{R} \times \mathbb{R}$ defined by

$$Zf(x,y) = \sum_{k=-\infty}^{\infty} f(x+k)e^{2\pi i y k}.$$

A striking property of the Zak transform, independently shown by Zak [3] and Janssen [6], is that Zf has a zero whenever Zf is continuous on $\mathbb{R} \times \mathbb{R}$. Actually, in certain special cases like when f is the Gaussian, this follows from elementary properties of theta series.

Now, the notion of the Zak transform admits a natural generalization to locally compact abelian groups (see Section 3). Given a locally compact abelian group G, its dual group $\widehat{G}$ and a uniform lattice K in G, the Zak transform, associated to K, of $f \in L^2(G)$ can be defined (almost everywhere) on $G \times \widehat{G}$ by

$$Zf(x,\omega) = \sum_{k \in K} f(xk)\omega(k).$$

The main purpose of this note is to extend the above result to compactly generated locally compact abelian groups. In fact, we are going to establish the following stronger result.

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Theorem. *Let G be a compactly generated locally compact abelian group with non-compact connected component of the identity, K a uniform lattice in G and Γ the annihilator of K in the dual group $\widehat{G}$ of G. Suppose that $g : G \times \widehat{G} \to \mathbb{C}$ is a continuous function satisfying the quasi-periodicity relation*

$$g(xk, \omega\gamma) = \overline{\omega(k)} g(x, \omega)$$

for all $(x, \omega) \in G \times \widehat{G}$ and $(k, \gamma) \in K \times \Gamma$. Then g has a zero.

The converse to the theorem also holds (Remark 1 in Section 4). As an immediate consequence of the theorem we obtain the following:

Corollary. *Let G and K be as in the theorem and Z the associated Zak transform. Let $f \in L^2(G)$ and suppose that Zf is continuous on $G \times \widehat{G}$. Then Zf has a zero.*

It is worthwhile to point out that conversely the corollary implies the theorem at least when G is first countable (compare Remark 5).

The proof of the theorem will be given in Section 2, whereas in Section 3 we deal with the Zak transform. Finally, in Section 4 we conclude with some remarks.

2. Proof of the theorem

Let G be an arbitrary locally compact abelian group and let K be a *uniform lattice* in G, that is, a discrete subgroup of G with compact quotient group G/K. In the sequel, Γ will denote the annihilator of K in $\widehat{G}$,

$$\Gamma = A(K, \widehat{G}) = \{\gamma \in \widehat{G} : \gamma(k) = 1 \text{ for all } k \in K\}.$$

Then Γ is a uniform lattice in $\widehat{G}$ since Γ is topologically isomorphic to $\widehat{G/K}$ and $\widehat{G}/\Gamma$ is topologically isomorphic to $\widehat{K}$ (via the restriction map $\omega\Gamma \to \omega|K$). The following lemma is required in the proof of the theorem.

Lemma 1. *Let $\mathcal{H}$ be a downward directed system of compact subgroups of G (with normalized Haar measures) such that $\bigcap_{H \in \mathcal{H}} H = \{e\}$. Let g be a continuous function on $G \times \widehat{G}$ such that*

$$g(xk, \omega\gamma) = \overline{\omega(k)} g(x, \omega)$$

for all $(x, \omega) \in G \times \widehat{G}$ and $(k, \gamma) \in K \times \Gamma$. For each $H \in \mathcal{H}$, define g_H on $G \times \widehat{G}$ by

$$g_H(x, \omega) = \int_H g(xh, \omega) dh.$$

Then g_H is continuous and satisfies $g_H(xk, \omega\gamma) = \overline{\omega(k)} g_H(x, \omega)$. If every g_H has a zero, then g has a zero.

Proof. That g_H is continuous follows immediately from the uniform continuity of g on compact subsets of $G \times \widehat{G}$. Moreover, for $(x, \omega) \in G \times \widehat{G}$ and $(k, \gamma) \in K \times \Gamma$,

$$g_H(xk, \omega\gamma) = \int_H g(xkh, \omega) dh = \overline{\omega(k)} \int_H g(xh, \omega) dh = \overline{\omega(k)} g_H(x, \omega).$$

Now suppose that every g_H has a zero. Since G/K and $\widehat{G}/\Gamma$ are compact, there exist compact subsets C of G and Δ of $\widehat{G}$ such that $G = CK$ and $\widehat{G} = \Delta\Gamma$. Due to the quasi-periodicity, for each $H \in \mathcal{H}$ there exist $x_H \in C$ and $\omega_H \in \Delta$ such that $g_H(x_H, \omega_H) = 0$. C and Δ being compact, by passing to a subnet if necessary, we can assume that $x_H \to x$ and $\omega_H \to \omega$ for some $x \in C$ and $\omega \in \Delta$. Finally, employing the uniform continuity of g on compact sets once more, we obtain that

$$|g(x,\omega)| = |g(x,\omega) - g_H(x_H,\omega_H)|$$

$$= \Big| \int_H \Big(g(x,\omega) - g(x_H h, \omega_H) \Big) dh \Big| \leq \int_H |g(x,\omega) - g(x_H h,\omega_H)| dh,$$

which converges to zero as $H \to \{e\}$. □

We now turn to the proof of the theorem. Notice first that by the structure theorem for compactly generated locally compact abelian groups [5, Theorem 9.8], G is of the form $G = \mathbb{R}^p \times \mathbb{Z}^q \times C$ where C is a compact group and $p \geq 1$ since by hypothesis G has a non-compact connected component of the identity. Now, compact groups are projective limits of Lie groups [10, p.99]. Therefore, there exists a system $\mathcal{H}$ of closed subgroups H of C as in Lemma 1 such that C/H is a Lie group for every $H \in \mathcal{H}$. Thus, for each $H \in \mathcal{H}$, there is a closed subgroup L_H of C such that $H \subseteq L_H, L_H$ is of finite index in C and $L_H/H = \mathbb{T}^{r_H}$ for some $r_H \in \mathbb{N}_0$.

By Lemma 1, for any such H, g_H is continuous, and once we have established that g_H has a zero on $G \times \widehat{G}$, it follows that g has a zero as well. To that end, fix H and set $L = L_H$ and $r = r_H$. Replacing g_H by g, we can therefore assume that g is constant on cosets of H. Let $\pi : G \to G/H$ denote the quotient homomorphism. Then $\pi(K) = KH/H$ is a uniform lattice in G/H, and

$$A(\pi(K), \widehat{G/H}) = \{\chi \in \widehat{G/H} : \chi \circ \pi \in A(K, \widehat{G})\}.$$

Now, the function $\tilde{g} : G/H \times \widehat{G/H} \to \mathbb{C}$ defined by

$$\tilde{g}(\pi(x), \chi) = g(x, \chi \circ \pi),$$

$x \in G$, $\chi \in \widehat{G/H}$, is continuous and satisfies the equation

$$\tilde{g}(\pi(x)\pi(k), \chi\,\delta) = \tilde{g}(\pi(x), \chi)\overline{(\chi \circ \pi)(k)}$$

for all $x \in G$, $k \in K$, $\chi \in \widehat{G/H}$ and $\delta \in A(\pi(K), \widehat{G/H})$. It suffices to show that $\tilde{g}$ has a zero. Thus, after moving to G/H, we can assume that $L = \mathbb{T}^r$. Towards a contradiction, suppose that $g(x,\omega) \neq 0$ for all $(x,\omega) \in G \times \widehat{G}$.

In what follows, for $x \in G$ and $\omega \in \widehat{G}$, let $x_1 \in \mathbb{R}^p$ and $\omega_1 \in \widehat{\mathbb{R}^p}$ denote the first component of x and ω, respectively. When convenient, we shall identify $\mathbb{R}^p$ with $\widehat{\mathbb{R}^p}$ by writing $\omega_1(x_1) = \exp 2\pi i \langle x_1, \omega_1 \rangle$. Let 1_C be the trivial character of C and $e_r : \mathbb{R}^r \to \mathbb{T}^r$ the covering homomorphism given by

$$e_r(u) = (e^{2\pi i u_1}, \dots, e^{2\pi i u_r})$$

for $u = (u_1, \dots, u_r) \in \mathbb{R}^r$. We define homomorphisms

$$\varphi_1 : \mathbb{R}^p \times \mathbb{R}^r \to \mathbb{R}^p \times \{0\} \times \mathbb{T}^r \subseteq G, \quad (x_1, u) \to (x_1, 0, e_r(u))$$

and

$$\varphi_2 : \widehat{\mathbb{R}^p} \times \widehat{\mathbb{R}^q} \to \widehat{\mathbb{R}^p} \times \widehat{\mathbb{Z}^q} \times \{1_C\} \subseteq \widehat{G}, \quad (\omega_1, \chi) \to (\omega_1, \chi|_{\mathbb{Z}^q}, 1_C).$$

Since g is continuous and has no zero on $G \times \widehat{G}$, we can consider the continuous function

$$(x_1, u, \omega_1, \chi) \to \frac{g(\varphi_1(x_1,u), \varphi_2(\omega_1,\chi))}{|g(\varphi_1(x_1,u), \varphi_2(\omega_1,\chi))|}$$

on $S = \mathbb{R}^p \times \mathbb{R}^r \times \widehat{\mathbb{R}^p} \times \widehat{\mathbb{R}^q}$. Since S is simply connected, there exists a continuous function $\varphi : S \to \mathbb{R}$ such that

$$\exp 2\pi i \varphi(x', \omega') = \frac{g(\varphi_1(x'), \varphi_2(\omega'))}{|g(\varphi_1(x'), \varphi_2(\omega'))|}$$

for all $x' \in \mathbb{R}^p \times \mathbb{R}^r$ and $\omega' \in \widehat{\mathbb{R}^p} \times \widehat{\mathbb{R}^q}$. Since φ_1 and φ_2 are homomorphisms, the quasi-periodicity relation for g implies that

$$\exp 2\pi i[\varphi(x'k', \omega') - \varphi(x', \omega')] = \overline{\varphi_2(\omega')(\varphi_1(k'))}$$

$$= \overline{\omega_1(k_1)} = \exp(-2\pi i \langle \omega_1, k_1 \rangle)$$

for all $(x', \omega') \in S$ and $k' \in \varphi_1^{-1}(K)$, and

$$\exp 2\pi i[\varphi(x', \omega'\gamma') - \varphi(x', \omega')] = 1$$

for all $(x', \omega') \in S$ and $\gamma' \in \varphi_2^{-1}(\Gamma)$.

Since S is connected and φ is continuous, it follows that given k' and γ', there are integers $m_1(k')$ and $m_2(\gamma')$ such that

(1) $\varphi(x'k', \omega') - \varphi(x', \omega') + \langle k_1, \omega_1 \rangle = m_1(k')$

and

(2) $\varphi(x', \omega'\gamma') - \varphi(x', \omega') = m_2(\gamma')$

for all $x' \in \mathbb{R}^p \times \mathbb{R}^r$ and $\omega' \in \widehat{\mathbb{R}^p} \times \widehat{\mathbb{R}^q}$. Applying (1) first and then (2) yields

$$\begin{aligned} \varphi(x'k', \omega'\gamma') &= \varphi(x', \omega'\gamma') - \langle k_1, \omega_1 \rangle - \langle k_1, \gamma_1 \rangle + m_1(k') \\ &= \varphi(x', \omega') + m_2(\gamma') - \langle k_1, \omega_1 \rangle - \langle k_1, \gamma_1 \rangle + m_1(k'). \end{aligned}$$

On the other hand, applying (1) and (2) in the reverse order gives

$$\begin{aligned} \varphi(x'k', \omega'\gamma') &= \varphi(x'k', \omega') + m_2(\gamma') \\ &= \varphi(x', \omega') - \langle k_1, \omega_1 \rangle + m_1(k') + m_2(\gamma'). \end{aligned}$$

Subtracting these two equations shows that

$$\langle k_1, \gamma_1 \rangle = 0$$

for all pairs (k_1, γ_1) such that $(k_1, 0, k_3) \in K$ for some $k_3 \in \mathbb{T}^r$ and $(\gamma_1, \gamma_2, 1_C) \in \Gamma$ for some $\gamma_2 \in \widehat{\mathbb{Z}^q}$.

We are now going to show that this is impossible. Notice first that, since $G' = \mathbb{R}^p \times L$ is open in G, $G'/(G' \cap K)$ is topologically isomorphic to $G'K/K \subseteq G/K$, which is compact. Hence $K \cap G'$ is cocompact in G'. Let K_1 denote the set of first components of elements in $K \cap G'$. Then K_1 contains a vector space basis for $\mathbb{R}^p$. Indeed, otherwise

$$K \cap G' \subseteq K_1 \times L \subseteq V \times L \subseteq \mathbb{R}^p \times L = G'$$

for some proper subspace V of $\mathbb{R}^p$, which contradicts the fact that $G'/(K \cap G')$ is compact.

Thus it only remains to verify that there exist $\gamma_1 \in \widehat{\mathbb{R}^p}$ and $\gamma_2 \in \widehat{\mathbb{Z}^q}$ such that $\gamma_1 \neq 0$ and $(\gamma_1, \gamma_2, 1_C) \in \Gamma$. Assume that $(\gamma_1, \gamma_2, 1_C) \in A(K, \widehat{G})$ only if $\gamma_1 = 0$. Then

$$A(KC, \widehat{G}) \subseteq A(\mathbb{R}^p \times C, \widehat{G}),$$

and hence $KC \supseteq \mathbb{R}^p \times C$, whence

$$K/(K \cap C) = KC/C \supseteq \mathbb{R}^p,$$

which is impossible since K is discrete. This finishes the proof of the theorem.

The idea of writing $g/|g|$, when possible, as the exponential of some continuous function occurs already in the proofs that Zak [3] and Janssen [6] gave for the existence of a zero in the case $G = \mathbb{R}$. For a different proof compare [1, p.18].

3. The Zak transform and zeros

If G is a locally compact abelian group and K a uniform lattice in G, then a *fundamental domain* for K will mean a Borel subset S of G such that every $x \in G$ can be uniquely written in the form $x = sk$ where $s \in S$ and $k \in K$.

Generalizing the classical notion of the Zak transform for the uniform lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$, we are going to introduce the Zak transform on $L^2(G)$ associated to K. The first step is to guarantee the existence of a fundamental domain for K.

Lemma 2. *Let G be a locally compact abelian group and K a uniform lattice in G. Then there exists a relatively compact fundamental domain for K.*

Proof. We assume first that G is compactly generated. Since G is a projective limit of second countable groups [10, p.104] and K is discrete, there exists a compact subgroup C of G such that $C \cap K = \{e\}$ and G/C is second countable. By [8, Lemma 1.1] there exists a relatively compact fundamental domain Q for KC/C in G/C. Let $q : G \to G/C$ denote the quotient homomorphism, and set $S = q^{-1}(Q)$. Clearly, S is a relatively compact Borel set, and using the fact that $K \cap C = \{e\}$, it is easy to check that S is indeed a fundamental domain for K.

Now, drop the assumption that G is compactly generated and choose an open compactly generated subgroup H of G. Since $K \cap H$ is a uniform lattice in H, by the preceding paragraph there exists a relatively compact fundamental domain S for $K \cap H$ in H. As H is open and G/K is compact, KH has finite index in G. Let F be a coset representative system for KH in G, and let $T = FS$. Then T is a relatively compact Borel set, and as above it is straightforward to verify that T is a fundamental domain for K in G. □

By Lemma 2 there exist relatively compact fundamental domains S for K in G and Ω for Γ in $\widehat{G}$.

Let the Haar measure on G be normalized so that Weil's formula holds, if we take on G/K the normalized Haar measure and the counting measure on K. Clearly, if G is σ-compact (equivalently, K is countable), then S has positive measure ($|S| > 0$). However, this is also true in the general case. To see this, choose a compactly generated open subgroup H of G containing S and observe that $Sk \cap H \neq \emptyset$ if and only if $k \in H$. Since H is σ-compact and K is discrete, there are only countably many such k. Thus H is a countable union of sets Sk, $k \in K$, whence $|S| > 0$.

The map $\Phi : S \to G/K$, $x \to xK$ is a continuous bijection. For each measurable subset M of S and with χ_M the characteristic function of M, Weil's formula gives

$$|M| = \int_G \chi_M(x)dx = \int_{G/K} \left(\sum_{k \in K} \chi_M(xk) \right) d(xK) = |\Phi M|.$$

Hence Φ maps the measure on S induced by the Haar measure on G to the normalized Haar measure on G/K.

Similarly, normalizing the Haar measures on $\widehat{G}$ and $\widehat{G}/\Gamma$ appropriately, the mapping $\Omega \to \widehat{G}/\Gamma$, $\omega \to \omega\Gamma$ transforms the induced measure on Ω into the Haar measure on $\widehat{G}/\Gamma$, and $|\Omega| = 1$.

The next lemma will show that for an arbitrary locally compact abelian group the Zak transform can be defined as indicated in the introduction.

Lemma 3. *Retain the preceding assumptions and notations, and let $f \in L^2(G)$. Then, for almost all $(x, \omega) \in S \times \Omega$,*

$$Zf(x,\omega) = \sum_{k\in K} f(xk)\omega(k)$$

converges, and the function Zf belongs to $L^2(S \times \Omega)$ and satisfies $\|Zf\|_2 = \|f\|_2$.

Proof. For $k \in K$, define $f_k \in L^2(S \times \Omega)$ by $f_k(x,\omega) = f(xk)\omega(k)$. Then

$$\sum_{k\in K} \|f_k\|_2^2 = \sum_{k\in K} \int_S \int_\Omega |f_k(x,\omega)|^2 d\omega dx = \sum_{k\in K} \int_S |f(xk)|^2 dx = \|f\|_2^2.$$

We claim that $\langle f_k, f_l\rangle = 0$ for $k, l \in K, k \neq l$. To show this recall that if C is a compact abelian group and φ a non-trivial character of C, then $\int_C \varphi(y)dy = 0$ [5, Lemma 23.19]. Applying this to $C = \widehat{G}/\Gamma$ and the character φ defined by

$$\varphi(\omega\Gamma) = \omega(kl^{-1}), \quad \omega \in \widehat{G},$$

we obtain

$$\int_\Omega \omega(kl^{-1})d\omega = \int_{\widehat{G}/\Gamma} \varphi(\omega\Gamma)d(\omega\Gamma) = 0,$$

and this in turn implies

$$\langle f_k, f_l\rangle = \int_S \int_\Omega f(xk)\overline{f(xl)}\omega(kl^{-1})d\omega dx = 0.$$

It follows that the series $\sum_{k\in K} f_k$ converges in $L^2(S \times \Omega)$ and satisfies

$$\|\sum_{k\in K} f_k\|_2^2 = \sum_{k\in K} \|f_k\|_2^2 = \|f\|_2^2.$$

In particular, $Zf(x,\omega)$ exists for almost all $(x,\omega) \in S \times \Omega$. □

We can now define the Zak transform Zf for $f \in L^2(G)$. Notice first that for every $(k,\gamma) \in K \times \Gamma$ and any finite subset H of K,

$$\sum_{h\in H} f(xkh)(\omega\gamma)(h) = \overline{\omega(k)} \sum_{l\in H} f(xl)\omega(l).$$

Thus $Zf(xk, \omega\gamma)$ converges if and only if $Zf(x,\omega)$ does. It follows from Lemma 3 that

$$Zf(x,\omega) = \sum_{k\in K} f(xk)\omega(k)$$

is defined for locally almost all $(x,\omega) \in G \times \widehat{G}$ (and, in fact, for a.a. (x,ω) if G is σ-compact), and this function is called the *Zak transform* of f. We say that Zf *is continuous on* $G \times \widehat{G}$ if there exists a continuous function g on $G \times \widehat{G}$ which agrees with Zf locally a.e. on $G \times \widehat{G}$. Of course, such a function g then satisfies the quasi-periodicity relation $g(xk, \omega\gamma) = \overline{\omega(k)}g(x,\omega)$ for all $(x,\omega) \in G \times \widehat{G}$ and $(k,\gamma) \in K \times \Gamma$. Hence an application of the theorem yields the corollary.

4. Some remarks

We finish the paper by adding remarks concerning the hypotheses of the theorem and some application.

Remark 1. The converse to the theorem also holds. That is, if G is a compactly generated locally compact abelian group, then G_0, the connected component of the identity, must be non-compact provided that G has the following property: For every uniform lattice K in G and $f \in L^2(G)$, Zf has a zero whenever Zf is continuous.

In fact, suppose that G_0 is compact so that $G = D \times C$ where D is discrete and C is compact. Choosing $K = D$ and $f = \chi_C$, one obtains for $x = dc$, $d \in D$, $c \in C$, and $\omega \in \widehat{G}$,

$$Zf(x, \omega) = \sum_{k \in D} f(xk)\omega(k) = \overline{\omega(d)}.$$

This formula shows that Zf is continuous and of modulus 1.

Remark 2. In general, a locally compact abelian group G need not contain a uniform lattice. The following example was kindly communicated by the referee.

Suppose G is the group $(\mathbb{Z}_4)^\infty$ with the topology obtained when the subgroup C generated by all elements of order 2 is declared to be open and compact. Then every discrete subgroup K of G has to be finite. Indeed, $K \cap C$ is finite and $x \to x^2$ is a homomorphism from K into $K \cap C$ with kernel $K \cap C$.

However, if G is of the form $G = \mathbb{R}^p \times D \times C$, where D is discrete and C is compact, then we can take $K = \mathbb{Z}^p \times D$. More specifically, if G is compactly generated, say $G = \mathbb{R}^p \times \mathbb{Z}^q \times C$, then an abundance of uniform lattices can be constructed as follows. Let h_1 be a homomorphism of $\mathbb{Z}^p \subseteq \mathbb{R}^p$ into C and let h_2 and h_3 be homomorphisms of $\mathbb{Z}^q$ into $\mathbb{R}^p$ and C, respectively. Then

$$K = \{(x_1 + h_2(x_2), x_2, h_1(x_1) + h_3(x_2)) : x_1 \in \mathbb{Z}^p, x_2 \in \mathbb{Z}^q\}$$

is a uniform lattice in G.

Remark 3. The condition that Zf be continuous is satisfied whenever f is continuous and rapidly decreasing outside of compact subsets of G. More precisely, it is well-known that if f is a continuous function on $\mathbb{R}^d$ such that $|f(x)| \leq c(1+\|x\|_2)^{-\alpha}$ for some $\alpha > 1$ and $c > 0$, then Zf is continuous. Slightly more general, it is not difficult to see that for $G = \mathbb{R}^p \times \mathbb{Z}^q \times C \subseteq \mathbb{R}^p \times \mathbb{R}^q \times C$, a similar hypothesis with respect to the $\mathbb{R}^p$ and $\mathbb{R}^q$ variables is sufficient.

For the two final remarks, let K denote a uniform lattice in the locally compact abelian group G, Γ the annihilator of K in $\widehat{G}$ and Z the Zak transform associated to K.

Remark 4. Let S and Ω be relatively compact fundamental domains for K in G and Γ in $\widehat{G}$, respectively. We have seen (Lemma 3) that, after suitably normalizing Haar measures, Z maps $L^2(G)$ unitarily into $L^2(S \times \Omega)$. It can be shown that Z is surjective provided that the mappings $S \to G/K$ and $\Omega \to \widehat{G}/\Gamma$ induce Hilbert space isomorphisms $L^2(S) \to L^2(G/K)$ and $L^2(\Omega) \to L^2(\widehat{G}/\Gamma)$, (compare the proof for $G = \mathbb{R}^d$ in [2]). This latter condition is satisfied if $S \to G/K$ and $\Omega \to \widehat{G}/\Gamma$ are Borel isomorphisms, that is, if S and Ω arise from Borel cross-sections $G/K \to G$

and $\widehat{G}/\Gamma \to \widehat{G}$. Now, the existence of such cross-sections is guaranteed when G (and hence $\widehat{G}$) is second countable ([8, Lemma 1.1] and [9, Theorem 4.2]).

Remark 5. Let G be a first countable compactly generated locally compact abelian group, and let K be a uniform lattice in G and Γ the annihilator of K in $\widehat{G}$. Choose relatively compact Borel sets S in G and Ω in $\widehat{G}$ such that the quotient mappings are Borel isomorphisms (see Remark 4). Suppose that g is a continuous function on $G \times \widehat{G}$ satisfying the quasi-periodicity relation $g(xk, \omega\gamma) = \overline{\omega(k)}\, g(x, \omega)$ for all $(x, \omega) \in G \times \widehat{G}$ and $(k, \gamma) \in K \times \Gamma$. Then, since $Z : L^2(G) \to L^2(S \times \Omega)$ is surjective, there exists $f \in L^2(G)$ such that $Zf = g$ a.e. on $S \times \Omega$, hence a.e. on $G \times \widehat{G}$. Thus, in this situation, the theorem and the corollary are equivalent.

Remark 6. Let $f \in L^2(G)$, and for $k \in K$ and $\gamma \in \Gamma$ define $\varphi_{k,\gamma} \in L^2(G)$ by $\varphi_{k,\gamma}(x) = \gamma(x) f(k^{-1}x)$. The collection of all these functions is called the *Gabor system associated with* f. In the classical situation, $G = \mathbb{R}^d$, the question of when this Gabor system forms a frame (an exact frame, an orthonormal basis) for $L^2(G)$ has been a matter of great interest.

In this context the Zak transform plays an important role. For instance, the set $\{\varphi_{k,\gamma} : k \in K, \gamma \in \Gamma\}$ constitutes a frame for $L^2(\mathbb{R}^d)$ with frame bounds A and B precisely when $A \leq |Zf| \leq B$ almost everywhere on $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ (see [2, Theorem 3.16]). Now, the proofs of these results carry over, in a straightforward manner, to a general locally compact abelian group G provided that the mapping $Z : L^2(G) \to L^2(S \times \Omega)$ is onto. By the preceding remark we know this to be true for suitable S and Ω at least when G is second countable.

In particular, from the corollary we can draw the following conclusion. Suppose that G is a first countable compactly generated locally compact abelian group with non-compact connected component of the identity. If $f \in L^2(G)$ is such that Zf is continuous, then the functions $\varphi_{k,\gamma}$ do not form a frame for $L^2(G)$.

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