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ON SELF-INTERSECTIONS OF IMMERSED SURFACES

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ABSTRACT. A daisy graph is a union of immersed circles in 3-space which intersect only at the triple points. It is shown that a daisy graph can always be realized as the self-intersection set of an immersed closed surface in 3-space and the surface may be chosen to be orientable if and only if the daisy graph has an even number of edges on each immersed circle.

It is well known that a general position immersion of a surface in euclidean 3-space may have two types of self-intersections: *double points* and *triple points*. If the surface is closed, then its self-intersection set is a union of immersed circles, called the *transverse components*, which intersect only at the triple points where three arcs meet in the same way as the three axes of 3-space meet at the origin. We call such a union of immersed circles a *daisy graph* which is so named since it can be obtained from copies of the *daisy* by adding bands; see Figure 1. For our purposes, we sometimes ignore the extra configurations at the triple points and consider a daisy graph as a union of embedded circles and a standard graph in 3-space with vertices being the triple points which are of degree 6, hence the notions of *vertices* and *edges* from graph theory make sense for a daisy graph and its transverse components. But observe that embedded circles that do not pass through triple points are considered to have zero number of edges.

The study of immersed surfaces in 3-space via their self-intersections goes back as far as Dehn [8] and Whitney [12]. Banchoff [1] showed that the number of triple points of an immersed closed surface in 3-space is congruent modulo two to the Euler characteristic of the surface. The proof works in any 3-manifold provided the image of the surface is null-homologous. More generally, Carter and Ko proved a congruence $\chi(f(M)) = \chi(M) + T(f)$ modulo two, where χ denotes the Euler number of a cell complex, $T(f)$ is the number of triple points of an immersion f of a closed surface M into an arbitrary 3-manifold N; see [4]. Their techniques could be applied to give a proof of the result of Izumiya and Marar [10] that

$$\chi(f(M)) = \chi(M) + T(f) + B(f)/2,$$

where $B(f)$ is the number of branched points, and now f is a general position map. This last result is an equality rather than a congruence. Carter [2] and Csikós and Szücs [7] addressed the problem of extending an immersion of a circle in a surface to a proper immersion of a surface in a 3-manifold bounded by the surface.

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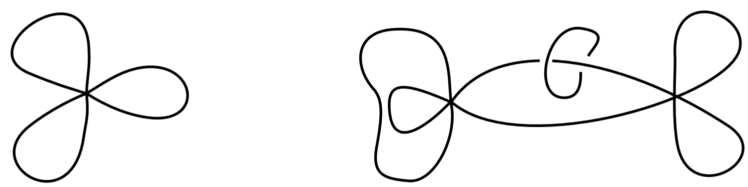

(a) A daisy (b) A daisy graph

FIGURE 1

Giller [9] showed that the problem of lifting an immersed surface in 3-space to an embedded surface in 4-space can be determined by examining its self-intersection set; see also Carter and Saito [5]. Cromwell and Marar [6] examined the possible generic surfaces that have the coordinate axes in 3-space as their self-intersections; see also Carter [3].

In this paper we shall prove the following:

Theorem. *A daisy graph in 3-space can always be realized as the self-intersection set of an immersed closed surface in 3-space and the surface may be chosen to be orientable if and only if each transverse component of the daisy graph has an even number of edges.*

Banchoff's theorem asserts that an immersed closed oriented surface in 3-space always has an even number of triple points; while our result indicates that among the daisy graphs with an even number of triple points, many can only be realized as the self-intersection sets of connected sums of copies of the Klein bottle, which are non-orientable. Figure 1(b) is an example of a daisy graph with two triple points and three transverse components having zero, one, and five edges respectively, which is not the self-intersection set of any immersed closed oriented surface.

It is worth remarking that if a daisy graph G has an even number of edges on each transverse component, then the number of triple points of G must be even. To see this, let u_k be the number of vertices of degree $2k$ on a transverse component T of G where $k = 1, 2, 3$, and let v_1 be the number of vertices of degree 2 of T which are vertices of degree 4 of other transverse components of G. Then the number of edges of T is $u_1 + 2u_2 + 3u_3$, which is even by the assumption. The assertion now follows since the number of triple points of G is the sum of $v_1 + (u_1 - v_1)/3 + u_3$ over all transverse components T, which is obviously even.

Proof of the Theorem. Let G be a daisy graph in 3-space. Then there exists an immersed compact surface X_G with boundary in a regular neighborhood of G, called a *cross-surface* for G, such that the self-intersection set of X_G is G. To be more precise, we let X_G be an embedded $D(3)$ restricted to a small 3-ball surrounding each triple point and be embedded $D(2) \times I$'s and $D(2) \times S^1$'s outside the 3-balls where

$$D(n) = \{(x_1, \ldots, x_n) \in \mathbf{R}^n | -1 \leq x_1, \ldots, x_n \leq 1,\ x_1 \cdots x_n = 0\}.$$

Then the self-intersection set of the resulting immersed surface X_G is G; see also Giller [2]. Figure 2(a) illustrates how the construction works near two adjacent triple points in which two embedded $D(3)$'s are connected via an embedded $D(2) \times I$.

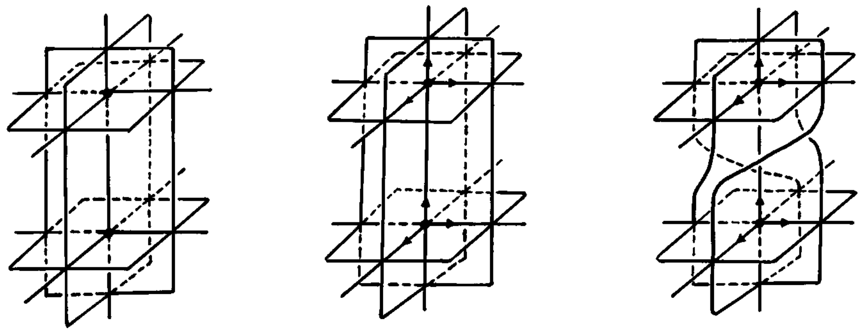

FIGURE 2

Notice that the cross-surfaces so constructed are not unique, which depend on how the embedded $D(2) \times I$'s are glued with the embedded $D(3)$'s and are often non-orientable. To get an oriented cross-surface, it is convenient to introduce the notion of an *arrowed daisy graph*. This is a daisy graph such that each triple point is equipped with three *arrows* which start from the triple points and end at points lying in the interiors of edges of G and such that only one arrow appears in a transverse arc passing through a triple point. Let the two 2-disks or annuli in each embedded $D(2) \times I$ or $D(2) \times S^1$ be arbitrarily oriented, and let the three 2-disks in each embedded $D(3)$ be oriented such that the three arrows coincide with the normal directions of the 2-disks at that triple point. Suppose that the embedded $D(2) \times I$'s are glued with the embedded $D(3)$'s in an oriented way as in Figure 2(b). Then the result is an oriented cross-surface for G.

Our interest in cross-surfaces stems from the fact that if a daisy graph G is the self-intersection set of an immersed closed surface M, then M restricts to a cross-surface X_G for G in a regular neighborhood of G whose boundary bounds the part of M outside the regular neighborhood of G. Hence a daisy graph is the self-intersection set of an immersed closed oriented surface if and only if it admits an oriented cross-surface X_G whose boundary bounds an embedded oriented surface missing X_G. This is equivalent to saying that ∂X_G represents the zero class in the first integral homology group of the complement of X_G in 3-space, which by Alexander duality is equivalent to saying that ∂X_G has a vanishing integral linking number with each loop in X_G, or equivalently, with each loop in G. Similarly, a daisy graph G is the self-intersection set of an immersed closed surface which is not necessarily orientable if and only if it admits a cross-surface X_G which is not necessarily orientable such that ∂X_G has a vanishing modulo two linking number with each loop in G.

To prove the theorem, we begin with an arrowed daisy graph G in 3-space and let X_G be an oriented cross-surface for G constructed as above. Push the interior of X_G along a normal vector field that restricts to the arrows at each triple point. One gets then an immersed surface $\overline{X}_G$ with $\partial \overline{X}_G = \partial X_G$ so that $\overline{X}_G$ intersects G only at the ends of arrows and at the end of each arrow there is exactly one intersection. Notice that each edge of G has at most two arrows, and in case there are two arrows these give rise to opposite orientations of the edge. It follows that the immersed surface $\overline{X}_G$ intersects each edge of G in at most two points, and in case there are two intersections these should have opposite sign.

Suppose that $\overline{X}_G$ intersects each edge of G in zero or two points. Then it follows by the discussion above that ∂X_G has a vanishing integral linking number with each loop in G. Hence G can be realized as the self-intersection set of an immersed oriented closed surface. Suppose now that there exists an edge E of G that intersects $\overline{X}_G$ in exactly one point. Then the part of X_G near E looks like the one shown in Figure 2(b), which one may modify as in Figure 2(c) to get a non-orientable cross-surface for G. After being pushed off along the arrows at each triple point, such a cross-surface is seen to have the same intersections with $G - E$ as $\overline{X}_G$ but now intersect E in two points. Repeating this procedure, one gets eventually a cross-surface for G whose boundary has a vanishing modulo two linking number with each loop in G. Hence the daisy graph G can be realized as the self-intersection set of an immersed closed surface in 3-space, which completes the proof of the first assertion of the theorem.

To prove the second assertion of the theorem, suppose that a daisy graph G is the self-intersection set of an immersed oriented closed surface and let X_G be the corresponding oriented cross-surface. Then ∂X_G has a vanishing integral linking number with each loop in G, hence with each transverse component of G. Arrow the daisy graph G according to the given orientation of X_G. Then the resulting arrowed daisy graph G has an even number of arrows on each transverse component by the discussion above. But this could happen only if each transverse component of G has an even number of edges since the number of arrows on a transverse component of an arrowed daisy graph is the same as the number of edges. On the other hand, suppose that each transverse component of G has an even number of edges. Then one may label the edges on each transverse component succesively by $0, 1, 0, 1, \ldots, 0, 1$. Rearrow the daisy graph G in such a way that there exist no arrow on edges numbered by 0 and two arrows on edges numbered by 1, and denote by Y_G the oriented cross-surface for G constructed using these arrows. Then the corresponding immersed surface $\overline{Y}_G$ intersects each edge of G in zero or two points. Hence G can be realized as the self-intersection set of an immersed closed oriented surface in 3-space. ☐

Remark 1. That a daisy graph can be realized as the self-intersection set of an immersed closed surface can alternatively be proved by performing surgery to copies of Boy's surface which is an immersed projective plane in 3-space with the daisy as the self-intersection set [1]. Since a daisy graph can always be obtained by adding bands to copies of the daisy, this is equivalent to saying that if two daisy graphs G_1 and G_2 can be realized as the self-intersection sets of two immersed closed surfaces which we denote by M_1 and M_2 respectively, then so is the band-summing $G = G_1 \#_b G_2$ via a band b. To see this, push the two ends of the core arc of b slightly off G_1 and G_2 within M_1 and M_2 respectively, and make the resulting arc α intersecting M_1 and M_2 transversely. Add a 1-handle connecting M_1 to M_2 with α as the core. One gets an immersed surface M with self-intersection set a disjoint union of G_1, G_2 and a collection $c_1, \ldots, c_m$ of circles. These circles may be ordered in such a way that there exist disjoint arcs $l_0, l_1, \ldots, l_m$ in M that meet the self-intersections of M only at the ends such that l_0 connects G_1 to c_1, l_m connects c_m to G_2, and l_k connects c_k to c_{k+1} for $1 \leq k \leq m-1$. Add 1-handles to M with the l_k the cores as in Banchoff [1] to connect G_1, G_2 and the c_k. The construction results in a non-orientable immersed closed surface with G as the self-intersection set.

Remark 2. The referee suggested yet another proof of the first assertion and the *if* part of the second assertion of our theorem which proceeds as follows. Replace the embedded $D(2) \times I$'s used in the construction of the cross-surface X_G for a daisy graph G by the embedded $\{\text{figure eight}\} \times I$'s and leave the structure of $D(3)$ at each triple point for the time being; such a construction appeared also in the work of Koschorke [11] and Csikós and Szücs [7]. In the neighborhood of a triple point, the construction results in one of the seven Cromwell-Marar surfaces described in [6]. In order to find a closed surface with G as its self-intersection set, we can twist the figure eights as indicated in Figure 2(c) until the configuration at a triple point is the one at which there are four triangles on the bounding sphere of the regular neighborhood of the triple point; this is the Cromwell-Marar configuration that gives the Roman surface. Then it is easy to cap off these triangles obtaining a closed non-orientable surface with G as its self-intersection set. If we want to preserve the orientability, the figure eight construction should be performed in an oriented way using an arrowed daisy graph as before. Then the boundary of the resulting oriented surface sitting on the bounding sphere of the regular neighborhood of each triple point consists of three circles, among which two can be capped off, while the third bounds a 2-disk that intersects G at the ends of the three arrows at the triple point. So if each transverse component of G has an even number of edges, then we can rearrow G as in our original proof and add 1-handles to eliminate pairwisely the intersection points of the 2-disks bounded by the third circles mentioned above with G to get an immersed oriented closed surface with G as the self-intersection set.

It should be noticed that the arguments in the two remarks above work for daisy graphs in an arbitrary 3-manifold. Hence a daisy graph in a 3-manifold can always be realized as the self-intersection set of an immersed closed surface, and if each transverse component of the daisy graph has an even number of edges, then the immersed surface may be chosen to be orientable. Nevertheless, since the evenness of the number of edges on each transverse component seems to be not necessary for a daisy graph in an arbitrary 3-manifold to be the self-intersection set of an immersed oriented closed surface unless the 3-manifold is a homology 3-sphere, the following general question still remains open.

Question. Let X be an arbitrary 3-manifold. Which daisy graphs, or more generally, which arrowed daisy graphs can be realized as the self-intersection set of an immersed closed oriented surface?

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