

ON COMPLETE METRIC SPACES CONTAINING THE SIERPIŃSKI CURVE

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ABSTRACT. It is proved that a complete metric space topologically contains the Sierpiński universal plane curve if and only if it has a subset with so-called *bypass property*, i.e. it has a subset K containing an arc such that for each $a \in K$ and for each open arc $A \subset K$ with $a \in A$, there exists an arbitrary small arc in $K \setminus \{a\}$ joining the two components of $A \setminus \{a\}$.

Investigating homogeneous continua which contain an arc, a characterization of complete metric spaces that topologically contain the Sierpiński universal plane curve has been obtained (Theorem 1 below). To formulate this result we need the following definition.

Definition 1. *A metric space X has the bypass property if*

- (1) *X contains an arc, and*
- (2) *for every arc xy in X and every point $z \in xy \setminus \{x, y\}$ and $\varepsilon > 0$ there is an arc $x'y'$ in $X \setminus \{z\}$ with diameter $< \varepsilon$ joining the two components of $xy \setminus \{z\}$.*

Theorem 1. *A complete metric space X topologically contains the Sierpiński universal plane curve if and only if X contains a subset with the bypass property.*

The aim of this note is to prove this theorem. Observe that sets with the bypass property need not be ‘nice’. They may be of the first category in themselves and they may have degenerate components or arc components. They may be ‘poor’, i.e. composed of countably many arcs in each boundary in the set. The theorem is a tool to conclude that complete metric spaces containing such a ‘poor’ subset are ‘rich’ in some sense—they topologically contain all planar one-dimensional spaces.

The theorem is related to the results of [3].

The paper is ended with a corollary on homogeneous continua and with three related questions.

To prove Theorem 1 we need the following lemma.

Lemma 2. *Let X have the bypass property. For every connected graph $G \subset X$ without end points and for every $\varepsilon > 0$ there is a connected graph $H \subset X$ without end points such that*

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- (a) H is the union of G and a finite number of arcs of diameter $< \varepsilon$ each intersecting G , and
- (b) for any arc $xy \subset G$ and any point $z \in xy \setminus \{x, y\}$ there is an arc in $H \setminus \{z\}$ joining x, y and contained in the ε -neighborhood of xy .

Proof. By the bypass property of X for each ε -small arc L in G and for each of its points z (except its end points) there is an ε -small arc $A_{L,z}$ in $X \setminus \{z\}$ joining the two components of $L \setminus \{z\}$. (Of course we can assume that no vertex of G belongs to $A_{L,z}$.) Let $U_{L,z}$ stand for an open arc satisfying $z \in U_{L,z} \subset \overline{U_{L,z}} \subset L \setminus A_{L,z}$. Observe that $A_{L,z}$ joins the two components of $L \setminus \{z'\}$ for any $z' \in U_{L,z}$. We will say that the arc $A_{L,z}$ corresponds to $U_{L,z}$.

Of course $U_{L,z}$ is a neighborhood of z in L , and also in G if z is not a vertex of G .

For each vertex v of G fix a finite collection $\{U_{L_1,v}, U_{L_2,v}, \dots, U_{L_{i_v},v}\}$ defined as above such that each pair of adjacent edges at v is joined by some $U_{L_k,v}$.

The set $G \setminus \bigcup_{k=1}^{i_v} U_{L_k,v} : v \text{ is a vertex of } G$ contains no vertex and it is compact, hence there is a finite cover $\{U_{L_{z_1},z_1}, U_{L_{z_2},z_2}, \dots, U_{L_{z_m},z_m}\}$ of it.

Let (for simplicity) U_1, U_2, \dots, U_n denote all these sets of this cover and all the above mentioned sets $U_{L_k,v}$ for all vertices v in G . Denote by A_1, A_2, \dots, A_n the arcs of the form $A_{L,z}$ which correspond to sets U_1, U_2, \dots, U_n .

Observe that the union $H = G \cup A_1 \cup A_2 \cup \dots \cup A_n$ has all the desired properties except one; this union may not be a graph.

If it is the case, it may be improved inductively. Suppose that $G \cup A_1 \cup A_2 \cup \dots \cup A_i$ is a graph but $G \cup A_1 \cup A_2 \cup \dots \cup A_i \cup A_{i+1}$ is not a graph. Then the subset $A_{i+1} \setminus (G \cup A_1 \cup A_2 \cup \dots \cup A_i)$ of the arc A_{i+1} has infinitely many components. Almost all of them may be replaced by finitely many small arcs in $G \cup A_1 \cup A_2 \cup \dots \cup A_i$ so that some arc A'_{i+1} close to A_{i+1} is obtained satisfying:

- A'_{i+1} corresponds to the same set U_{i+1} as A_{i+1} , and
- $A'_{i+1} \setminus (G \cup A_1 \cup A_2 \cup \dots \cup A_i)$ has finitely many components.

Hence $G \cup A_1 \cup A_2 \cup \dots \cup A_i \cup A'_{i+1}$ is a graph. □

For any metric space X , any set $Y \subset X$ and for any $\varepsilon > 0$ define

$$T_\varepsilon(Y, X) = \bigcup \{A : A \text{ is an arc in } X, \text{ diam } A < \varepsilon \text{ and } A \cap Y \neq \emptyset\}.$$

Observe the following easy property of the operator T :

$$(1) \quad T_\alpha(T_\beta(Y, X), X) \subset T_{\alpha+\beta}(Y, X)$$

for all positive numbers α, β .

Construction of G_n 's. Fix any metric space B with the bypass property. Take any simple closed curve G_1 in B and put $\varepsilon_0 = 1$. We inductively construct an increasing sequence G_n of graphs in B and a sequence ε_n of positive numbers as follows.

Assume G_1, \dots, G_n and $\varepsilon_0, \dots, \varepsilon_{n-1}$ are already constructed. Let

$$\varepsilon_n < \min\{\varepsilon_{n-1}/2; 1/(8n)\}$$

be a fixed positive number such that for any two points $x, y \in G_n$ satisfying $d(x, y) < 5\varepsilon_n$ there exists an arc $A \subset G_n$ with $x, y \in A$ and $\text{diam } A < 1/(3n)$.

Let G_{n+1} be the graph H guaranteed by Lemma 2 for graph $G = G_n$ and for $\varepsilon = \varepsilon_n$. Let $F = \bigcup_{n \in \mathcal{N}} G_n$. □

The above construction plays a crucial role in the proof of the main result of the paper. Now, we come to prove some of its properties.

First, notice that $G_{n+1} = T_{\varepsilon_n}(G_n, G_{n+1}) \subset T_{\varepsilon_n}(G_n, F)$ by the construction of G_n 's (see Lemma 2(a)). Applying (1), by the induction we obtain

$$(2) \quad G_{n+k} = T_{2\varepsilon_n}(G_n, G_{n+k}) \subset T_{2\varepsilon_n}(G_n, F)$$

for each pair of positive integers n, k .

Property 3. *For any $\varepsilon > 0$ there exists a $\delta > 0$ such that for each triple of points $p, x, y \in F$ satisfying $x \neq p \neq y$ and $d(x, y) < \delta$, there exists an arc $A \subset F \setminus \{p\}$ with $x, y \in A$ and $\text{diam } A < \varepsilon$.*

Proof. Given any $\varepsilon > 0$, take a positive integer n such that $1/n < \varepsilon/2$, and put $\delta = \varepsilon_n$. Let points $p, x, y \in F$ satisfy $x \neq p \neq y$ and $d(x, y) < \delta$. Then there is a $k \in \mathcal{N}$ such that $x, y \in G_{n+k}$. By (2) there are points $a_x, a_y \in G_n$ and arcs A_x, A_y in G_{n+k} such that $x, a_x \in A_x$; $y, a_y \in A_y$ and $\text{diam } A_x, \text{diam } A_y < 2\varepsilon_n$. We have $d(a_x, a_y) < \text{diam } A_x + \delta + \text{diam } A_y < 5\varepsilon_n$, thus, by the definition of ε_n , there is an arc $A_1 \subset G_n$ such that $a_x, a_y \in A_1$ and $\text{diam } A_1 < 1/(3n)$. Since $\text{diam } A_x, \text{diam } A_y < 2\varepsilon_n < 1/(4n)$, then the diameter of the set $K = A_x \cup A_1 \cup A_y$ is $< 1/n < \varepsilon/2$. Next, K is a locally connected continuum in G_{n+k} , thus it contains an arc A in G_{n+k} with $x, y \in A$. If $p \in A$, we apply the construction of G_i 's and condition (b) of Lemma 2, to modify arc A and obtain an arc A' in G_{n+k+1} such that $x, y \in A' \subset G_{n+k+1} \setminus \{p\}$ and $\text{diam } A' < 2 \cdot 1/n < \varepsilon$. □

Recall that a metric space X is said to have property S provided for any $\varepsilon > 0$, space X may be represented as the finite union of its connected subsets each having diameter less than ε (see [5], Chapter I, §15).

Property 4. *Set F has property S .*

Indeed, let n satisfy $5\varepsilon_n < \varepsilon$. Since G_n is a graph, it can be represented as the finite union of arcs A_1, \dots, A_k each having diameter $< \varepsilon_n$. Hence $F = T_{2\varepsilon_n}(G_n, F) = T_{2\varepsilon_n}(A_1, F) \cup \dots \cup T_{2\varepsilon_n}(A_k, F)$ by (2). The reader can easily observe that the above union is that required. □

Theorem 5. *If a complete metric space X contains a subspace B with the bypass property, then X contains a locally connected curve with no local separating point.*

Proof. Assuming some extra condition, we repeat the construction of graphs G_1, G_2, \dots in B (see the Construction of G_n 's just below (1)), to obtain the closure of their union having the desired properties. In this construction we employ the fact that G_n 's are compact ANR's.

The modification is the following. On the initial step of the construction we take a neighborhood V_1 of G_1 in X , admitting a 1-retraction $r_1 : V_1 \rightarrow G_1$ (i.e. a retraction satisfying $\text{diam } r_1^{-1}(y) < 1$ for each $y \in G_1$).

On each inductive step we take ε_n additionally so small, that $N_{\varepsilon_n}(G_n) \subset V_n$. (The previous restrictions on ε_n are still satisfied.) Finally, we take any neighborhood $V_{n+1} \subset V_n$ of G_{n+1} in X , admitting an ε_n -retraction $r_{n+1} : V_{n+1} \rightarrow G_{n+1}$. The construction is complete.

Put $Z = \overline{F}$, where $F = \bigcup_{n \in \mathcal{N}} G_n$ and the closure is taken in X .

Of course Z is a closed connected set. Notice that $Z \subset \bigcap_{n \in \mathcal{N}} V_n$. Thus for each n set Z has ε_{n-1} -retraction $r_n|_Z$ onto graph G_n . Therefore, Z is one-dimensional.

By the construction F satisfies the conclusions of Properties 3 and 4.

Since F has property S by Property 4, its closure Z does also (see [5], Chapter I, §15, (15.3), p.20). Thus Z is locally connected. Moreover, again by property S , set Z is a totally bounded, complete metric subspace of X . Hence Z is compact.

One can easily verify that Property 3 for F implies that Z has no local separating point. \square

Proof of Theorem 1. By Theorem 5 space X contains a locally connected curve Z with no local separating point. If some open (in Z) subset of Z is embeddable in the plane, then this subset is homeomorphic to some open subset of the Sierpiński universal plane curve (see [6]). Otherwise Z is the Menger universal curve (see [1], [2] or [4], Th.4.11). In both cases Z contains (topologically) a copy of the Sierpiński universal plane curve.

The converse implication is obvious. \square

To show some applications of the result let us state the following definition.

Definition 2. A point p of a space X is said to be a local arc separating point in X provided there exists a neighborhood U of p in X and an arc $A \subset U$ such that $A \setminus \{p\}$ meets two different arc components of $U \setminus \{p\}$.

Observe that any space with the bypass property contains no local arc separating point.

By Theorem 1 and by the universal properties of the Sierpiński curve we obtain the following corollary on homogeneous continua.

Corollary 6. For any homogeneous continuum X containing an arc either X topologically contains all planar one-dimensional sets, or each point of X is a local arc separating point in X .

We end the paper with three questions connected with the bypass property and homogeneous continua. In these questions X denotes an arbitrary homogeneous continuum.

Question 1. If X contains arbitrarily small simple closed curves, does it topologically contain the Sierpiński curve?

Question 2. If X contains the Sierpiński curve, does X have the bypass property? (Equivalently, does X have no local arc separating point?)

Question 3. If X contains the Sierpiński curve, does X contain either two-dimensional disc, or the Menger universal curve?

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