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BOUNDEDNESS OF THE CESÀRO OPERATOR ON MIXED NORM SPACES

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ABSTRACT. In this note, the boundedness of the Cesàro operator on mixed norm space $H_{p,q}(\varphi)$, $0 < p, q \leq \infty$, is proved.

1. INTRODUCTION

Let $(a) = \{a_n\}_{n=1}^{\infty}$ is in l^p, $1 < p < \infty$, then the sequence

$$C[a] = \left\{ \frac{1}{n+1} \sum_{k=0}^{n} a_k \right\}_{n=0}^{\infty}$$

has l^p-norm satisfying

$$||C[a]||_p \leq \frac{p}{p-1} ||a||_p.$$

Thus C is a bounded linear operator on l^p for $1 < p < \infty$. This operator on l^p has been studied by many authors (see [1], [5]).

Let f be holomorphic on the unit disc U with Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, call

$$C[f](z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n$$

the *Cesàro operator* acting on f.

With the result of Hardy [3] on trigonometric series and M. Riesz's theorem, one can prove that the Cesàro operator $C[f]$ is bounded on $H^p(U)$ for $1 < p < \infty$, where $H^p(U)$ is the usual Hardy space. Siskakis [7] studied the spectrum of $C[f]$ on $H^p(U)$, as a by-product he obtained that $C[f]$ is bounded on $H^p(U)$, $1 \leq p < \infty$. Recently he [8] gave another proof of the boundedness of $C[f]$ on $H^1(U)$, independent of spectrum theory. After that Jie [4] proved that $C[f]$ is also bounded on $H^p(U)$, $0 < p < 1$. A natural question is whether the Cesàro operator $C[f]$ is bounded on Bergman space $L_a^p(U)$, $0 < p < \infty$, or weighted Bergman space $L_a^p((1-|z|^2)^{\alpha} dm(z))$, $\alpha > -1$, where dm is the Lebesgue measure on $\mathbf{C}$.

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In this note, we will consider the more general setting of, say, the mixed norm space $H_{p,q}(\varphi)$, and prove that $C[f]$ is bounded on $H_{p,q}(\varphi)$. As a particular case, $C[f]$ is bounded on weighted Bergman space $L_a^p((1-|z|^2)^\alpha dm(z))$, $\alpha > -1$, and so it is also bounded on Bergman space $L_a^p(U)$.

2. Main Theorem

A positive continuous function φ on [0,1) is normal, if there exist $0 < a < b$ such that

(i) $\frac{\varphi(r)}{(1-r)^a}$ is non-increasing in $[0,1)$ and $\lim_{r\to 1} \frac{\varphi(r)}{(1-r)^a} = 0$;

(ii) $\frac{\varphi(r)}{(1-r)^b}$ is non-decreasing in $[0,1)$ and $\lim_{r\to 1} \frac{\varphi(r)}{(1-r)^b} = \infty$.

For a normal function φ, $0 < p, q \leq \infty$, the holomorphic function f on the unit disc U is said to belong to the mixed norm space $H_{p,q}(\varphi)$, if

$$||f||_{p,q,\varphi} = \left\{ \int_0^1 (1-r)^{-1} \varphi^p(r) M_q^p(r,f) dr \right\}^{1/p} < \infty, \quad 0 < p < \infty,$$
$$||f||_{\infty,q,\varphi} = \sup_{0<r<1} \varphi(r) M_q(r,f) < \infty.$$

Here

$$M_q(r,f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^q d\theta \right\}^{\frac{1}{q}}, \qquad 0 < q < \infty,$$
$$M_\infty(r,f) = \sup_{0\leq\theta\leq 2\pi} |f(re^{i\theta})|.$$

For any $\alpha > -1$, $\varphi(r) = (1-r)^{(\alpha+1)/p}$ is a normal function. When $p = q$, $H_{p,p}((1-r^2)^{(\alpha+1)/p})$ is just the weighted Bergman space $L_a^p((1-|z|^2)^\alpha dm(z))$. In fact, this follows from the integral formula in polar coordinates

$$\int_U |f(z)|^p (1-|z|^2)^\alpha dm(z) = 2\pi \int_0^1 r(1-r^2)^\alpha M_p^p(r,f) dr$$

and the monotonicity of $M_p(r,f)$.

Our main result is the following theorem.

Theorem. *Let $0 < p, q \leq \infty$, φ be a normal function. Then the Cesàro operator $C[f]$ is bounded on $H_{p,q}(\varphi)$. In particular, $C[f]$ is bounded on $L_a^p((1-|z|^2)^\alpha dm(z))$, $0 < p < \infty$, $\alpha > -1$.*

3. Some lemmas

The following lemmas are needed in the proof of the theorem.

Lemma 1 ([2, p. 758])**.** *Let $1 \leq k < \infty$, $\mu > 0$, $\delta > 0$, $h : (0,1) \longrightarrow [0,\infty)$ be measurable. Then*

$$\int_0^1 (1-r)^{k\mu-1} \left\{ \int_0^r (r-t)^{\delta-1} h(t) dt \right\}^k dr \leq K \int_0^1 (1-r)^{k\mu+k\delta-1} h^k(r) dr.$$

Here, and latter, K denotes a positive constant, not necessarily the same at each occurrence.

From this lemma, we can derive the following key lemma.

Lemma 2. *Let $1 \leq k < \infty$, $\delta > 0$, $h : (0,1) \longrightarrow [0,\infty)$ be measurable, and φ be a normal function. Then*

$$\begin{aligned} &\int_0^1 (1-r)^{-1}\varphi^k(r)\left\{\int_0^r (r-t)^{\delta-1}h(t)dt\right\}^k dr \\ &\quad \leq K\int_0^1 (1-r)^{k\delta-1}\varphi^k(r)h^k(r)dr. \end{aligned} \tag{1}$$

Proof. Taking $\epsilon > 0$ sufficient small such that $\epsilon/k < a$, where a is the constant in the definition of the normal function φ. If we set $\mu = \epsilon/k$ and replace $h(t)$ by $\frac{\varphi(t)}{(1-t)^{\epsilon/k}}h(t)$ in Lemma 1, then we get the following

$$\begin{aligned} &\int_0^1 (1-r)^{\epsilon-1}\left\{\int_0^r (r-t)^{\delta-1}\frac{\varphi(t)}{(1-t)^{\epsilon/k}}h(t)dt\right\}^k dr \\ &\quad \leq K\int_0^1 (1-r)^{k\delta-1}\varphi^k(r)h^k(r)dr. \end{aligned} \tag{2}$$

On the other hand, since $\frac{\varphi(t)}{(1-t)^{\epsilon/k}}$ is non-increasing, it follows that

$$\begin{aligned} &\int_0^1 (1-r)^{-1}\varphi^k(r)\left\{\int_0^r (r-t)^{\delta-1}h(t)dt\right\}^k dr \\ &\quad \leq \int_0^1 (1-r)^{\epsilon-1}\left\{\int_0^r (r-t)^{\delta-1}\frac{\varphi(t)}{(1-t)^{\epsilon/k}}h(t)dt\right\}^k dr. \end{aligned} \tag{3}$$

Combining (2) and (3), we get the desired inequality (1).

Lemma 3 ([6, Lemma 8])**.** *Let $h(t)$ be a positive non-decreasing continuous function of t. Then for any $0 < u < v < \infty$ and $0 < r < 1$, the following inequality holds*

$$\left\{\int_0^1 (1-t)^{v-1}h^v(rt)dt\right\}^{1/v} \leq K\left\{\int_0^1 (1-t)^{u-1}h^u(rt)dt\right\}^{1/u}.$$

Lemma 4. *(i) If $1 \leq q \leq \infty$, $0 < \delta < 1$, $0 < r \leq 1$, then*

$$r^\delta M_q(r, C[f]) \leq K\int_0^r (r-\rho)^{\delta-1}h_1(\rho)d\rho, \tag{4}$$

where $h_1(\rho) = \frac{1}{(1-\rho)^\delta}M_q(\rho, f)$.

(ii) If $1 \leq q \leq \infty$, $0 < p < 1$, $0 < r \leq 1$, then

$$r^p M_q^p(r, C[f]) \leq K\int_0^r (r-\rho)^{p-1}h_2(\rho)d\rho, \tag{5}$$

where $h_2(\rho) = \frac{1}{(1-\rho)^p}M_q^p(\rho, f)$.

(iii) If $0 < q < 1$, $0 < \delta < q$, $0 < r \leq 1$, then

$$r^\delta M_q^q(r, C[f]) \leq K\int_0^r (r-\rho)^{\delta-1}h_3(\rho)d\rho, \tag{6}$$

where $h_3(\rho) = \frac{1}{(1-\rho)^\delta}M_q^q(\rho, f)$.

(iv) If $0 < q < 1$, $0 < p < q$, $0 < \delta < p$, $0 < r \leq 1$, then

$$r^\delta M_q^p(r, C[f]) \leq K \int_0^r (r-\rho)^{\delta-1} h_4(\rho) d\rho, \tag{7}$$

where $h_4(\rho) = \frac{1}{(1-\rho)^\delta} M_q^p(\rho, f)$.

Proof. A direct computation with power series gives

$$C[f](z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n = \int_0^1 \frac{f(tz)}{1-tz} dt.$$

(i) First assume $1 \leq q < \infty$. Minkowsky's inequality shows that

$$\begin{aligned} M_q(r, C[f]) &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^1 \frac{f(tre^{i\theta})}{1-tre^{i\theta}} dt \right|^q d\theta \right\}^{1/q} \\ &\leq \int_0^1 \frac{1}{1-tr} M_q(tr, f) dt \\ &\leq \int_0^1 \frac{1}{(1-t)^{1-\delta}} \frac{1}{(1-tr)^\delta} M_q(tr, f) dt \\ &\leq \frac{1}{r^\delta} \int_0^r (r-\rho)^{\delta-1} h_1(\rho) d\rho. \end{aligned} \tag{8}$$

This proves (4).

When $q = \infty$, note that

$$|C[f](re^{i\theta})| \leq \int_0^1 \frac{|f(tre^{i\theta})|}{|1-tre^{i\theta}|} dt \leq \int_0^1 \frac{M_\infty(tr, f)}{1-tr} dt,$$

then

$$M_\infty(r, C[f]) \leq \int_0^1 \frac{1}{1-tr} M_\infty(tr, f) dt. \tag{9}$$

The remaining proof is the same as above.

(ii) We have proved in (i) that

$$M_q(r, C[f]) \leq \int_0^1 \frac{1}{1-tr} M_q(tr, f) dt.$$

Taking $u = p, v = 1$ and $h(t) = \frac{1}{1-t} M_q(t, f)$ in Lemma 3 implies

$$\begin{aligned} M_q(r, C[f]) &\leq \int_0^1 \frac{1}{1-tr} M_q(tr, f) dt \\ &\leq K \left\{ \int_0^1 (1-t)^{p-1} \frac{1}{(1-tr)^p} M_q^p(tr, f) dt \right\}^{1/p} \\ &= K \frac{1}{r} \left\{ \int_0^r (r-\rho)^{p-1} h_2(\rho) d\rho \right\}^{1/p} \end{aligned}$$

This proves (5).

(iii) Let $0 < q < 1$, in [4] the following inequality is proved:

$$M_q^q(r, C[f]) \leq K \int_0^1 (1-t)^{q-1} M_q^q(tr, g) dt, \tag{10}$$

where $g(z) = \frac{f(z)}{1-z}$. Since

$$M_q^q(tr, g) = \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(tre^{i\theta})|^q}{|1-tre^{i\theta}|^q}d\theta \le \frac{1}{(1-tr)^q}M_q^q(tr, f),$$

we have

$$\begin{aligned} M_q^q(r, C[f]) &\le \int_0^1 \frac{(1-t)^{q-1}}{(1-tr)^q}M_q^q(tr, f)dt \\ &= K\int_0^1 \frac{(1-t)^{q-1}}{(1-tr)^{q-\delta}}\frac{1}{(1-tr)^\delta}M_q^q(tr, f)dt \\ &\le K\int_0^1 (1-t)^{\delta-1}h_2(tr)dt \\ &= \frac{K}{r^\delta}\int_0^r (r-\rho)^{\delta-1}h_2(\rho)d\rho. \end{aligned}$$

This proves (6).

(iv) Let $0 < q < 1$, $0 < p < q$. It follows from (10) and Lemma 3 that

$$\begin{aligned} M_q(r, C[f]) &\le K\left\{\int_0^1 (1-t)^{q-1}M_q^q(tr, g)dt\right\}^{1/q} \\ &\le K\left\{\int_0^1 (1-t)^{p-1}M_q^p(tr, g)dt\right\}^{1/p} \\ &\le K\left\{\int_0^1 \frac{(1-t)^{p-1}}{(1-tr)^p}M_q^p(tr, f)dt\right\}^{1/p} \\ &= K\left\{\int_0^1 \frac{(1-t)^{p-1}}{(1-tr)^{p-\delta}}\frac{1}{(1-tr)^\delta}M_q^p(tr, g)dt\right\}^{1/p} \\ &\le K\left\{\int_0^1 (1-t)^{\delta-1}h_3(tr)dt\right\}^{1/p} \\ &\le K\left\{\frac{1}{r^\delta}\int_0^r (r-\rho)^{\delta-1}h_3(\rho)d\rho\right\}^{1/p}. \end{aligned}$$

This proves (7), and completes the proof of Lemma 4.

4. Proof of the theorem

Now we can give the proof of the theorem, which is divided into six cases.

Proof of the Theorem. Case 1. $1 \le q \le \infty, 1 \le p < \infty$.

Taking the transformation of variable $t = r^{p\delta+1}$ in the following inequality

$$\|C[f]\|_{p,q,\varphi}^p = \int_0^1 (1-t)^{-1}\varphi^p(t)M_q^p(t, C[f])dt, \tag{11}$$

due to the non-decreasing of the integral means and

$$\varphi(r^{p\delta+1}) = \frac{\varphi(r^{p\delta+1})}{(1-r^{p\delta+1})^b}(1-r^{p\delta+1})^b \le \frac{\varphi(r)}{(1-r)^b}(1-r^{p\delta+1})^b \le K\varphi(r),$$

we obtain

$$\begin{aligned}||C[f]||_{p,q,\varphi}^p &= (p\delta+1)\int_0^1 (1-r^{p\delta+1})^{-1}\varphi^p(r^{p\delta+1})M_q^p(r^{p\delta+1},C[f])r^{p\delta}dr\\ &\le K\int_0^1 (1-r)^{-1}\varphi^p(r)M_q^p(r,C[f])r^{p\delta}dr.\end{aligned}$$

Then using (4) and Lemma 2, we have

$$\begin{aligned}||C[f]||_{p,q,\varphi}^p &\le \int_0^1 (1-r)^{-1}\varphi^p(r)\left\{\int_0^r (r-\rho)^{\delta-1}h_1(\rho)d\rho\right\}^p dr\\ &\le K\int_0^1 (1-r)^{-1}\varphi^p(r)M_q^p(r,f)dr\\ &= K||f||_{p,q,\varphi}^p.\end{aligned}$$

Case 2. $1\le q\le\infty,\quad 0<p<1.$

Change the variable by $t=r^{p+1}$ in the right side of (11) and use the same approach as in Case 1 to get

$$||C[f]||_{p,q,\varphi}^p \le K\int_0^1 (1-r)^{-1}\varphi^p(r)M_q^p(r,C[f])r^p dr.$$

Taking $\delta=p$, $k=1$, and replacing $\varphi(r)$ by $\varphi^p(r)$, $h(t)$ by $h_2(t)$ in Lemma 2, we obtain

$$\begin{aligned}||C[f]||_{p,q,\varphi}^p &\le K\int_0^1 (1-r)^{-1}\varphi^p(r)\left\{\int_0^r (r-\rho)^{p-1}h_2(\rho)d\rho\right\}dr\\ &\le K\int_0^1 (1-r)^{-1}\varphi^p(r)M_q^p(r,f)dr = K||f||_{p,q,\varphi}^p.\end{aligned}$$

Case 3. $0<q<1,\quad p\ge q.$

In this case $p/q\ge 1$. Take the integral transformation $t=r^{\frac{p}{q}\delta+1}$ in the right side of (11), to get

$$||C[f]||_{p,q,\varphi}^p \le K\int_0^1 (1-r)^{-1}\varphi^p(r)M_q^p(r,C[f])r^{\frac{p}{q}\delta}dr.$$

It follows from (6) and Lemma 2 that

$$\begin{aligned}||C[f]||_{p,q,\varphi}^p &\le K\int_0^1 (1-r)^{-1}\left(\varphi^q(r)\right)^{p/q}\left(M_q^q(r,C[f])r^{\delta}\right)^{p/q}dr\\ &\le K\int_0^1 (1-r)^{-1}\left(\varphi^q(r)\right)^{p/q}\left\{\int_0^r (r-\rho)^{\delta-1}h_3(\rho)d\rho\right\}^{p/q}dr\\ &\le K\int_0^1 (1-r)^{\frac{p}{q}\delta-1}\varphi^p(r)(h_3(r))^{p/q}dr\\ &= K\int_0^1 (1-r)^{-1}\varphi^p(r)M_q^p(r,f)dr\\ &= K||f||_{p,q,\varphi}^p.\end{aligned}$$

Case 4. $0 < q < 1, \quad p < q$.

Taking the integral transformation $t = r^{\delta+1}$ in the right side of (11) and using (7) yields

$$\begin{aligned}
||C[f]||^p_{p,q,\varphi} &\le K \int_0^1 (1-r)^{-1} \varphi^p(r) M_q^p(r, C[f]) r^\delta dr \\
&\le K \int_0^1 (1-r)^{-1} \varphi^p(r) \left\{ \int_0^r (r-\rho)^{\delta-1} h_4(\rho) d\rho \right\} dr \\
&\le K \int_0^1 (1-r)^{\delta-1} \varphi^p(r) h_4(r) dr \\
&= K \int_0^1 (1-r)^{-1} \varphi^p(r) M_q^p(r, f) dr = K ||f||^p_{p,q,\varphi}.
\end{aligned}$$

Case 5. $p = \infty, 1 \le q \le \infty$.

We claim that for $0 < u < \infty$,

$$\int_0^1 \frac{(1-t)^{u-1}}{\varphi^u(tr)(1-tr)^u} dt \le \frac{K}{\varphi^u(r)}. \tag{12}$$

In fact,

$$\int_0^1 \frac{(1-t)^{u-1}}{\varphi^u(tr)(1-tr)^u} dt \le K \frac{(1-r)^{au}}{\varphi^u(r)} \int_0^1 \frac{(1-t)^{u-1}}{(1-tr)^{u+au}} dt \le \frac{K}{\varphi^u(r)}.$$

Here we have used the inequality [6, p. 625]

$$\int_0^1 \frac{(1-t)^{u-1}}{(1-tr)^{u+au}} dt \le \frac{K}{(1-r)^{au}}.$$

It follows from the claim, (8) and (9) that

$$\begin{aligned}
M_q(r, C[f]) &\le \int_0^1 \frac{1}{1-tr} M_q(rt, f) dt \\
&\le K \sup_{0<t<1} (\varphi(rt) M_q(rt, f)) \int_0^1 \frac{1}{(1-tr)\varphi(tr)} dt \\
&\le K \sup_{0<s<1} (\varphi(s) M_q(s, f)) \frac{1}{\varphi(r)}.
\end{aligned}$$

Namely

$$\varphi(r) M_q(r, C[f]) \le K \sup_{0<s<1} \varphi(s) M_q(s, f) = K ||f||_{\infty,q,\varphi}.$$

This proves $||C[f]||_{\infty,q,\varphi} \le K ||f||_{\infty,q,\varphi}$.

Case 6. $p = \infty, 0 < q < 1$.

From (10) and the claim, we have

$$\begin{aligned}
M_q^q(r, C[f]) &\le K \int_0^1 \frac{(1-t)^{q-1}}{(1-tr)^q} M_q^q(tr, f) dt \\
&\le K \sup_{0<s<1} (\varphi^q(s) M_q^q(s, f)) \int_0^1 \frac{(1-t)^{q-1}}{\varphi^q(tr)(1-tr)^q} dt \\
&\le K ||f||^q_{\infty,q,\varphi} \frac{K}{\varphi^q(r)}.
\end{aligned}$$

Namely $||C[f]||_{\infty,q,\varphi} \le K ||f||_{\infty,q,\varphi}$. This completes the proof of the Theorem.

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