PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 12, December 1998, Pages 3571–3579
S 0002-9939(98)04550-X

THE EXPOSED POINTS OF THE SET OF INVARIANT MEANS ON AN IDEAL

TIANXUAN MIAO

(Communicated by J. Marshall Ash)

Abstract. Let G be a σ-compact locally compact nondiscrete group and let Q be a G-invariant ideal of $L^\infty(G)$. We denote the set of left invariant means m on $L^\infty(G)$ that are zero on Q (i.e. $m(f) = 0$ for all $f \in Q$) by LIM_Q. We show that, when G is amenable as a discrete group and the closed G-invariant subset of the spectrum of $L^\infty(G)$ corresponding to Q is a G_δ-set, LIM_Q is very large in the sense that every nonempty G_δ-subset of LIM_Q contains a norm discrete copy of $\beta\mathbb{N}$, where $\beta\mathbb{N}$ is the Stone-Čech compactification of the set $\mathbb{N}$ of positive integers with the discrete topology. In particular, we prove that LIM_Q has no exposed points in this case and every nonempty G_δ-subset of the set of left invariant means on $L^\infty(G)$ contains a norm discrete copy of $\beta\mathbb{N}$.

1. Introduction and notations

Let G be a locally compact group with a fixed left Haar measure λ. If G is compact, we assume $\lambda(G) = 1$. For $f \in L^\infty(G)$ and $x \in G$, the left translation of f by x is defined by ${}_xf(y) = f(xy)$, $y \in G$. A left invariant mean on $L^\infty(G)$ is a positive linear functional on $L^\infty(G)$ with $m(1) = 1$ and $m({}_xf) = m(f)$ for all $x \in G$ and all $f \in L^\infty(G)$. We say that G is amenable if there exists a left invariant mean on $L^\infty(G)$. The set of left invariant means on $L^\infty(G)$ is denoted by LIM (see [4], [7] and [8] for details about amenable groups). For a subset Ω of G, the characteristic function of Ω is denoted by 1_Ω and the complement of Ω in G is denoted by $\Omega^c = G \sim \Omega$.

It is well known that $L^\infty(G)$ is a commutative Banach algebra under pointwise multiplication. If $\mathcal{S}$ is the spectrum of $L^\infty(G)$, then $L^\infty(G) = C(\mathcal{S})$ by the Gelfand isomorphism. The left action of G on $L^\infty(G)$ induces an action of G on $\mathcal{S}$; consequently corresponding to any closed G-invariant ideal Q of $L^\infty(G)$ is a closed G-invariant subset $\tilde{Q}$ of $\mathcal{S}$ where $\tilde{Q} = \{\, \theta \in \mathcal{S} : \theta(f) = 0 \text{ for all } f \in Q \,\}$ and vice versa. We denote the set of left invariant means m on $L^\infty(G)$ that are zero on Q (i.e. $m(f) = 0$ for all $f \in Q$) by LIM_Q. Then LIM_Q is a convex subset of LIM and it can be identified with the set of probability measures on $\tilde{Q}$ that are invariant

Received by the editors December 12, 1996 and, in revised form, April 20, 1997.
1991 *Mathematics Subject Classification.* Primary 43A07.
Key words and phrases. Locally compact groups, amenable groups, invariant means, invariant ideals, exposed points.
This research is supported by an NSERC grant.

under the action of G. By a G_δ-subset of LIM_Q or LIM in this paper we mean a G_δ-subset of LIM_Q or LIM in the w^*-topology of $L^\infty(G)^*$.

Let C be a convex subset of LIM. An element $m_0 \in C$ is called an exposed point of C if there exists an $f_0 \in L^\infty(G)$ such that $Re\, m(f_0) < Re\, m_0(f_0)$ whenever $m \in C$ and $m \neq m_0$, where $Re\,\alpha$ denotes the real part of a complex number α. The set of exposed points of C is denoted by Exp C. Let $\beta\mathbb{N}$ be the Stone-Čech compactification of the set $\mathbb{N}$ of positive integers with the discrete topology. We say C contains a norm discrete copy of $\beta\mathbb{N}$ if there is a linear map $\psi : \ell^\infty(\mathbb{N})^* \to L^\infty(G)^*$ and a positive number α such that $\alpha\|\theta\| \leq \|\psi(\theta)\| \leq \|\theta\|$ for all $\theta \in \ell^\infty(\mathbb{N})^*$ and $\psi(\mathcal{F}) \subseteq C$, where

$$\mathcal{F} = \big\{\theta \in \ell^\infty(\mathbb{N})^* : \theta \geq 0,\ \|\theta\| = 1 \text{ and } \theta(f) = 0 \text{ if } f \in \ell^\infty(\mathbb{N}) \text{ with } \lim_n\ f(n) = 0\big\}.$$

It is known that $\beta\mathbb{N} \sim \mathbb{N} \subseteq \mathcal{F}$ and consequently card $\mathcal{F} = 2^c$.

In [1] Granirer asked whether LIM_Q has any exposed points (also see Talagrand [11]). The author proved in [5] that LIM has no exposed points if G is σ-compact and amenable as a discrete group. Talagrand [10] proved that LIM_Q is large in the sense that every nonempty G_δ-subset of LIM_Q contains a norm discrete copy of $\beta\mathbb{N}$ if G is a compact abelian group. Our main result of this paper is the following:

Main Theorem. *Let G be a σ-compact locally compact nondiscrete group. Let $\{f_n\}$ be any sequence in $L^\infty(G)$, and $\mathcal{A} = \{\, m \in LIM : \ m(f_n) = 0 \text{ for all } n\,\}$. If G is amenable as a discrete group, then $\mathcal{A}$ has no exposed points. If $\mathcal{A} \neq \varnothing$, then $\mathcal{A}$ contains a norm discrete copy of $\beta\mathbb{N}$.*

We will prove that every nonempty G_δ-subset of LIM contains a nonempty subset $\mathcal{A}$ as above. Hence we have the following corollary, which improves the main result we obtained in [5] (see section 2 for the statement and proofs of Corollaries 1–3).

Corollary 1. *Let G be a σ-compact locally compact nondiscrete group. If G is amenable as a discrete group, then every nonempty G_δ-subset of LIM contains a norm discrete copy of $\beta\mathbb{N}$.*

If $\tilde{Q}$ is a G_δ-subset of $\mathcal{S}$, we will show that LIM_Q is a G_δ-subset of LIM. So we have the following:

Corollary 3. *Let G be a σ-compact locally compact nondiscrete group. Let Q be a closed G-invariant ideal of $L^\infty(G)$. If G is amenable as a discrete group and $\tilde{Q}$ is a G_δ-subset of $\mathcal{S}$, then every nonempty G_δ-subset of LIM_Q contains a norm discrete copy of $\beta\mathbb{N}$. In particular, LIM_Q has no exposed points.*

Remarks. (1) Talagrand [10] proved that if G is a compact abelian group and Q is any closed G-invariant ideal of $L^\infty(G)$, then LIM_Q is large in the sense that every nonempty G_δ-subset of LIM_Q contains a norm discrete copy of $\beta\mathbb{N}$. We do not assume that G is abelian in our corollary above (note that abelian groups are amenable as discrete groups), but we do require that $\tilde{Q}$ be a G_δ-subset of $\mathcal{S}$.

(2) Our Corollary 2 of the Main Theorem is also an LIM version of Granirer's Theorem (see Paterson [7], (7.28), or Granirer [1]):

Theorem (Granirer)**.** *Let G be a noncompact, σ-compact locally compact group. Let $\{f_n\}$ be a sequence in $L^\infty(G)$ and $\mathcal{C}$ a compact, G_δ-subset of the spectrum of*

$L^\infty(G)$. Let

$$\mathcal{A} = \{m \in TLIM : supp\ (m) \subseteq \mathcal{C},\ m(f_n) = 0,\ n = 1, 2, 3, \dots\},$$

where $TLIM$ is the set of topologically invariant means on G (see [7]). If G is amenable as a discrete group, then $\mathcal{A}$ has no exposed points. If $\mathcal{A} \neq \varnothing$, then $\mathcal{A}$ is not separable.

By taking the real part of a function, we can assume, without loss of generality, all functions considered in this paper to be real valued.

2. The main results

To prove our main theorem, we need the following lemma. Its proof is a modification of the proof of Theorem 6D of Talagrand [9].

Lemma. *Let G be a σ-compact nondiscrete locally compact group and let $f_n \in L^\infty(G)$, $n = 1, 2, \dots$. Suppose $\mathcal{A} = \{\, m \in LIM : m(f_n) = 0,\ n = 1, 2, \dots\}$. If G is amenable as a discrete group and $\mathcal{A} \neq \phi$, then for any $f_0 \in L^\infty(G)$ and $\epsilon > 0$ there exists an open set $\Omega \subseteq G$ such that $\lambda(\Omega) < \epsilon$ and $\sup\{m(f_0) : m \in \mathcal{A}\} = \sup\{m(1_\Omega f_0) : m \in \mathcal{A}\}$.*

Proof. Step 1. We will prove that for any $\epsilon > 0$ and any $f \in L^\infty(G)$, there exist a $\mu \in LIM$ and an open set $\Omega \subseteq G$ such that

$$\lambda(\Omega) < \epsilon,\ \mu(1_\Omega) = 1 \text{ and } \mu(f) \geq a = \sup\{m(f) : m \in LIM\} - \epsilon.$$

For $u = (u_1, u_2, \dots, u_n) \in G^n$, let $C_u = \{t \in G : \frac{1}{n}\sum_{j=1}^{n} {}_{u_j}f(t) > a\}$. By the definition of the number a, $\lambda(C_u) > 0$. It follows from Lemma 6C of Talagrand [9] that there is an open set Ω_n of G with $\lambda(\Omega_n) < \epsilon\, 2^{-n-1}$ such that $\lambda(C_u \cap \bigcap_{i=1}^{q} t_i\Omega_n) > 0$ for all $u \in G^n$, q and $t_1, t_2, \dots, t_q \in G$. Put $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. Then $\lambda(\Omega) \leq \epsilon$ and, for all n, q and all $u \in G^n$ and $t \in G^q$, we have $\lambda(C_u \cap \bigcap_{i=1}^{q} t_i\Omega) > 0$. It follows from the proof of the step 1 of Theorem 6D in Talagrand [9] that such a μ as above exists.

Step 2. Let $0 < \epsilon < 1$ be fixed. Put

$$\mathcal{N}_\epsilon = \{\, m \in LIM : \text{ there is an open subset } \Omega \subseteq G \text{ such that } \lambda(\Omega) < \epsilon \text{ and } m(1_\Omega) = 1\,\}.$$

Let $t \in G$ and $m \in LIM$. If Ω is an open set of G with $m(1_\Omega) = 1$ and $\lambda(\Omega) < \epsilon$, then we have

$$1 \geq m(1_{\Omega \cap t\Omega}) \geq m(1_\Omega) - m(1_{t\Omega^c}) = 1 \Rightarrow m(1_{\Omega \cap t\Omega}) = 1.$$

If G is not compact, then $\lambda(\Omega \cap t\Omega) \to 0$ when "$t \to \infty$". If G is compact, then

$$\inf_{t \in G} \lambda(\Omega \cap t\Omega) \leq \int_G \lambda(\Omega \cap t\Omega)\, dt = \int_G 1_\Omega(x) 1_{t\Omega}(x)\, dx\, dt = \lambda(\Omega)^2 \leq \epsilon^2.$$

Therefore, for all $\mu \in \mathcal{N}_\epsilon$ and $\eta > 0$, there exists an open set Ω of G such that $\lambda(\Omega) < \eta$ and $\mu(1_\Omega) = 1$.

If $\mu_1, \mu_2 \in \mathcal{N}_\epsilon$, then there exist open sets Ω_1, Ω_2 of G with $\lambda(\Omega_1) < \frac{1}{2}\epsilon$ and $\lambda(\Omega_2) < \frac{1}{2}\epsilon$ such that $\mu_1(\Omega_1) = \mu_2(\Omega_2) = 1$. Hence, for any $0 \leq \delta \leq 1$,

$(\delta\mu_1 + (1-\delta)\mu_2)(1_{\Omega_1\cup\Omega_2}) = 1$ and $\lambda(\Omega_1 \cup \Omega_2) < \epsilon$. So $\delta\mu_1 + (1-\delta)\mu_2 \in \mathcal{N}_\epsilon$ and $\mathcal{N}_\epsilon$ is convex.

By the Hahn-Banach Theorem, $LIM \subseteq \bar{\mathcal{N}_\epsilon}^{w^*}$, where $\bar{\mathcal{N}_\epsilon}^{w^*}$ is the closure of $\mathcal{N}_\epsilon$ in the w^*-topology of $L^\infty(G)^*$. (Otherwise, let $m_0 \in LIM \sim \bar{\mathcal{N}}^{w^*}$, there exist numbers r_1 and r_2 with $r_1 < r_2$ and $f \in L^\infty(G)$ such that $m(f) < r_1 < r_2 < m_0(f)$ for all $m \in \mathcal{N}_\epsilon^{w^*}$. This contradicts Step 1).

Step 3. Let $\alpha = \sup\{\, m(f_0) : m \in \mathcal{A}\,\}$. Since $\mathcal{A}$ is w^*-closed, there exists a $\nu \in \mathcal{A}$ such that $\nu(f_0) = \alpha$ and $\nu(f_n) = 0$ for $n = 1, 2, \ldots$. Let $\epsilon > 0$ be given. For each n, let $\epsilon_n = \epsilon\, 2^{-n-1}$. By Step 2, $\mathcal{N}_{\epsilon_n}$ is w^*-dense in LIM. Hence

$$\mathcal{N}_{\epsilon_n} \bigcap \{\, m \in LIM : \, |m(f_k) - \nu(f_k)| < \frac{1}{n},\, k = 0, 1, 2, \ldots, n \,\} \neq \varnothing.$$

Take an element μ_n from this set. Then $|\mu_n(f_k) - \nu(f_k)| < \frac{1}{n}$ for $k = 0, 1, 2, \ldots, n$ and there exists an open subset $\Omega_n \subseteq G$ such that $\lambda(\Omega_n) < \epsilon\, 2^{-n-1}$ and $\mu_n(1_{\Omega_n}) = 1$. Put $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. Since LIM is compact in the w^*-topology of $L^\infty(G)^*$, there exists a cluster point μ of $\{\mu_n\}$ in the w^*-topology. Then $\lambda(\Omega) < \epsilon$, $\mu(1_\Omega) = 1$ and $\mu(f_k) = \nu(f_k) = 0$ for $k = 1, 2 \ldots$. Also, $\mu(f_0) = \nu(f_0) = \alpha$. Hence $\mu \in \mathcal{A}$ and

$$\alpha = \mu(f_0) = \mu(f_0 1_\Omega) + \mu(f_0 1_{\Omega^c}) = \mu(f_0 1_\Omega). \quad \square$$

In the proof of our Main Theorem below, we will use a Baire category argument to split a function into infinitely many functions in $L^\infty(G)$ such that each of them supports an invariant mean in $\mathcal{A}$; then we use this to embed $\mathcal{F}$ into $\mathcal{A}$ (see the following theorem for the definition of $\mathcal{A}$).

Main Theorem. *Let G be a σ-compact locally compact nondiscrete group. Let $\{f_n\}$ be any sequence in $L^\infty(G)$ and let*

$$\mathcal{A} = \{\, m \in LIM : \, m(f_n) = 0 \textit{ for all } \, n \,\}.$$

If G is amenable as a discrete group, then Exp $\mathcal{A} = \varnothing$. If $\mathcal{A} \neq \varnothing$, then $\mathcal{A}$ contains a norm discrete copy of $\beta\mathbb{N}$.

Proof. Let $G = \bigcup_{q=1}^{\infty} K_q$, where each K_q is a compact subset of G and $K_i \subseteq K_{i+1}$ for $i = 1, 2, \ldots$. We assume that $\{f_j\}$ is a dense subset of the linear span of $\{f_j\}$ in $L^\infty(G)$ without loss of generality. If $f \in L^\infty(G)$ and β is a real number,

$$\mathcal{A}_{f,\beta} = \{\, m \in LIM : \, m(f) = \beta \text{ and } m(f_n) = 0 \text{ for all } \, n \,\}.$$

Assume that $\mathcal{A} \neq \varnothing$. Let $g \in L^\infty(G)$ and let $\alpha = \sup\{m(g) : \, m \in \mathcal{A}\}$. Then there exists an $m_0 \in \mathcal{A}$ with $m_0(g) = \alpha$ by the w^*-compactness of $\mathcal{A}$. We will show that $\mathcal{A}_{g,\alpha}$ contains a norm discrete copy of $\beta\mathbb{N}$. Consequently, $\mathcal{A}$ has no exposed points.

To show that $\mathcal{A}$ contains a norm discrete copy of $\beta\mathbb{N}$, we first observe that, without loss of generality, for the set $\mathcal{A}_{g,\alpha}$ we may assume $0 \leq g \leq 1$ and $\alpha > 0$, since if we take

$$g_0 = \frac{1 + g + \|g\|_\infty}{\|1 + g + \|g\|_\infty\|_\infty} \quad \text{ and } \quad \alpha_0 = \sup\{m(g_0) \colon \, m \in \mathcal{A}\} = \frac{1 + \alpha + \|g\|_\infty}{\|1 + g + \|g\|_\infty\|_\infty},$$

then $\mathcal{A}_{g,\alpha} = \mathcal{A}_{g_0,\alpha_0}$ and $\alpha_0 > 0$. We can also assume that $\lambda\{t \in G : g(t) > 0\} < \infty$.

In fact, it follows from our Lemma above that there exists an open set Ω with $\lambda(\Omega) < 1$ and $\alpha = \sup\{m(g1_\Omega) : m \in \mathcal{A}\}$. Then $\mathcal{A}_{(g1_\Omega),\alpha} \subseteq \mathcal{A}_{g,\alpha}$ and $\lambda\{t \in G : (g1_\Omega)(t) > 0\} < \infty$. Therefore, we can assume that $\lambda\{t \in G : g(t) > 0\} < \infty$, $m(1_\Omega) = 1$ for all $m \in \mathcal{A}_{g,\alpha}$ and $supp f_n \subseteq \Omega$ without loss of generality (in fact, we can take a subset of $\mathcal{A}_{g,\alpha}$ consisting of all elements supported by Ω. So we consider $f_n 1_\Omega$ only).

Next, we show that for such a g and α we can find two functions g_1, g_2 and $m_1, m_2 \in \mathcal{A}_{g,\alpha}$ such that
(a) $0 \leq g_i \leq 1, i = 1, 2$;
(b) $m_i(g_i) = \alpha, i = 1, 2$;
(c) $g = g_1 + g_2$;
(d) $m_1(g_2) = m_2(g_1) = 0$.
Let $X = \{f \in L^\infty(G) : 0 \leq f \leq g\}$. Then $(X, \|\cdot\|_1)$ is a complete metric space. Let $l, n \in N$ and $n > 0$ be fixed. For each $p, q \in N$, put

$$X_{l,p,q,} = \Big\{ \ f \in X : \ \text{there exist } x_1, x_2, \ldots, x_p \in K_q \text{ and } 1 \leq j \leq l \\ \text{with } \lambda\{ \ t \in G : \frac{1}{p} \sum_{i=1}^{p} {}_{x_i}(f(t) + f_j(t)) > \alpha - \frac{1}{n}\} = 0 \ \Big\}.$$

Using a Baire category argument, we will show, in the next three steps, that there exists an $f \in X$ such that both f and $g - f$ are not in $X_{l,p,q,}$ for all l, p and q.

Step 1. First, we show each $X_{l,p,q}$ is closed. Suppose $h_k \in X_{l,p,q}$ and $h_k \to h$ in $(X, \|\cdot\|_1)$. By passing to a subsequence of $\{h_k\}$ if necessary we can assume that there exists $j \in \{1, 2, \ldots, l\}$ such that for each k there exist $x_1^k, x_2^k, \ldots, x_p^k \in K_q$ such that

$$\lambda\{t \in G : \frac{1}{p} \sum_{i=1}^{p} {}_{x_i^k}(h_k(t) + f_j(t)) > \alpha - \frac{1}{n}\} = 0.$$

Since K_q is compact, by passing to a subnet if necessary we may assume that, for each $1 \leq i \leq p$, $x_i^k \to x_i$. We claim $\lambda\{t \in G : \frac{1}{p} \sum_{i=1}^{p} {}_{x_i}(h(t) + f_j(t)) > \alpha - \frac{1}{n}\} = 0$. If not, let $\delta > 0$ be such that

$$\lambda\{t \in G : \frac{1}{p} \sum_{i=1}^{p} {}_{x_i}(h(t) + f_j(t)) > \alpha - \frac{1}{n} + \delta\} = \epsilon > 0.$$

Put

$$B = \{t \in G : \frac{1}{p} \sum_{i=1}^{p} {}_{x_i}(h(t) + f_j(t)) > \alpha - \frac{1}{n} + \delta\}.$$

Then we have

$$\begin{aligned}
&\int_G |\frac{1}{p}\sum_{i=1}^p {}_{x_i}(h(t)+f_j(t)) - \frac{1}{p}\sum_{i=1}^p {}_{x_i^k}(h_k(t)+f_j(t))|\,dt \\
&\leq \int_G |\frac{1}{p}\sum_{i=1}^p {}_{x_i}h(t) - \frac{1}{p}\sum_{i=1}^p {}_{x_i^k}h_k(t)|\,dt + \int_G |\frac{1}{p}\sum_{i=1}^p {}_{x_i}f_j(x) - \frac{1}{p}\sum_{i=1}^p {}_{x_i^k}f_j(x)|\,dt \\
&\leq \frac{1}{p}\sum_{i=1}^p \int_G |{}_{x_i}h - {}_{x_i^k}h|\,dt \\
&\qquad + \frac{1}{p}\sum_{i=1}^p \int_G |{}_{x_i^k}h - {}_{x_i^k}h_k|\,dt + \frac{1}{p}\sum_{i=1}^p \|{}_{x_i}f_j(t) - {}_{x_i^k}f_j(t)\|_1 \to 0
\end{aligned}$$

as $x_i^k \to x_i$ for each $1 \leq i \leq p$.

On the other hand, since for each $x \in B$,

$$|\frac{1}{p}\sum_{i=1}^p {}_{x_i}(h(t)+f_j(t)) - \frac{1}{p}\sum_{i=1}^p {}_{x_i^k}(h_k(t)+f_j(t))| \geq \delta$$

we have

$$\begin{aligned}
&\int_G |\frac{1}{p}\sum_{i=1}^p {}_{x_i}(h(t)+f_j(t)) - \frac{1}{p}\sum_{i=1}^p {}_{x_i^k}(h_k(t)+f_j(t))|\,dt \\
&\geq \int_B |\frac{1}{p}\sum_{i=1}^p {}_{x_i}(h(t)+f_j(t)) - \frac{1}{p}\sum_{i=1}^p {}_{x_i^k}(h_k(t)+f_j(t))|\,dt \\
&\geq \delta\lambda(B) \geq \delta\epsilon
\end{aligned}$$

for all k. This is a contradiction. Therefore our claim is proved and it follows that $h \in X_{l,p,q}$.

Step 2. Also, $X_{l,p,q}$ is nowhere dense. In fact, for any $f \in X$ and any $\epsilon > 0$, by our Lemma, there is a subset Ω of G and an $m \in \mathcal{A}_{g,\alpha}$ such that $\lambda(\Omega) < \epsilon$, $m(1_\Omega) = 1$ and $m(g1_\Omega) = \alpha$. Let $f^* = g1_\Omega + f1_{G\sim\Omega}$. Then $f^* \in X$ and $\|f^* - f\|_1 = \|g1_\Omega - f1_\Omega\|_1 < 2\epsilon$. Since $m(f^*) = m(g1_\Omega) = \alpha > \alpha - \frac{1}{n}$ and $m \in \mathcal{A}_{g,\alpha}$, for any $1 \leq j \leq l$,

$$\lambda\{t \in G : \frac{1}{p}\sum_{i=1}^p {}_{x_i}(f^*(t)+f_j(t)) > \alpha - \frac{1}{n}\} \neq 0 \text{ for all } (x_1, x_2, \ldots, x_p) \in K_q^p.$$

Hence $f^* \notin X_{l,p,q}$.

Step 3. For any $l,p,q \in N$, let $X^c_{l,p,q} = \{f \in X : g - f \in X_{l,p,q}\}$. Then $X_{l,p,q}$ and $X^c_{l,p,q}$ are isometric in $(X, \|\cdot\|_1)$. So $X^c_{l,p,q}$ is also closed and nowhere dense in $(X, \|\cdot\|_1)$. Hence there exists an $f \in X \sim \bigcup_{l,p,q} (X_{l,p,q} \cup X^c_{l,p,q})$ by the completeness of X.

Next we will show that there exist $m_1, m_2 \in \mathcal{A}_{g,\alpha}$ with $m_1(f) = m_2(g-f) = \alpha$. Let

$$H = \text{ linear span of } \{\ F - {}_tF :\ F \in L^\infty(G) \text{ and } t \in G\ \} \ \cup\ \{f_j\}.$$

For any $\epsilon > 0$, since $\{f_j\}$ is dense in the linear span of $\{f_j\}$ in $L^\infty(G)$, there exists an h in the linear span of $\{\ F - {}_tF:\ F \in L^\infty(G) \text{ and } t\ \in G\ \}$ and j such that

$$\inf_{s\in H} \operatorname*{ess\,sup}_{t\in G} (f(t) + s(t)) > \operatorname*{ess\,sup}_{t\in G} (f(t) + h(t) + f_j(t)) - \epsilon.$$

Since G is amenable as a discrete group, by Følner's condition, there exist a k and an $x = (x_1, x_2, \ldots, x_k) \in G^k$ such that $\|\frac{1}{k} \sum_{i=1}^{k} {}_{x_i}h\|_\infty < \epsilon$ (see Granirer [3] for its proof).

Since $G = \bigcup_{q=1}^{\infty} K_q$, there is a $q \in N$ such that $x_1, x_2, \ldots, x_k \in K_q$. Thus, since $f \notin X_{l,k,q}$ for all l, we have $\lambda\{t \in G : \frac{1}{k} \sum_{i=1}^{k} {}_{x_i}(f(t) + f_j(t)) > \alpha - \frac{1}{n}\} \neq 0$ for all j. By Proposition 5 in Granirer [2], we have

$$\begin{aligned}
\sup_{m\in\mathcal{A}} m(f) &= \inf_{s\in H} \operatorname*{ess\,sup}_{t\in G} (f(t) + s(t)) \\
&\geq \operatorname*{ess\,sup}_{t\in G} (f(t) + h(t) + f_j(t)) - \epsilon \\
&\geq \operatorname*{ess\,sup}_{t\in G} \frac{1}{k} \sum_{i=1}^{k} {}_{x_i}(f(t) + f_j(t)) - 2\epsilon \\
&\geq \alpha - \frac{1}{n} - 2\epsilon.
\end{aligned}$$

Therefore, there exists an $m_n \in \mathcal{A}$ such that $m_n(f) \geq \alpha - \frac{1}{n}$. Similarly, since for all $x_1, x_2, \ldots, x_p \in G$ and all j, $\lambda\{t \in G : \frac{1}{p} \sum_{i=1}^{p} {}_{x_i}((g-f)(t) + f_j(t)) > \alpha - \frac{1}{n}\} \neq 0$, there exists an $M_n \in \mathcal{A}$ such that $M_n(g - f) \geq \alpha - \frac{1}{n}$. Let m and M be w^* limit points of m_n and M_n, respectively. Then $m, M \in \mathcal{A}$, $m(f) \geq \alpha$ and $M(g-f) \geq \alpha$. Since $0 \leq f \leq g$ and $0 \leq g - f \leq g$, $m(g) \geq \alpha$ and $M(g) \geq \alpha$. Hence $m(f) = M(g-f) = \alpha$ and $m, M \in A_{g,\alpha}$. Therefore, if we take $g_1 = g - f$, $g_2 = f$, $m_1 = M$, $m_2 = m$, then $m_i(g_i) = \alpha$, $0 \leq g_i \leq 1$ and $m_i \in A_{g,\alpha}$ for $i = 1, 2$. We also have that $m_1(g_2) = 0$, $m_2(g_1) = 0$ and $g = g_1 + g_2$.

Now we are ready to embed $\mathcal{F}$ into $\mathcal{A}_{g,\alpha}$. By continuing the same argument as above inductively (replace f by g_2, and so on), we construct a sequence $\{g_n\}$ of functions in $L^\infty(G)$ with the following properties.

(1) $0 \leq g_n \leq g$ for all n and for any k, we have $0 \leq g_1 + g_2 + \cdots + g_k \leq g$.
(2) There exists a sequence $\{m_n\}$ in $A_{g,\alpha}$ such that $m_n(g_n) = \alpha$ and $m_n(g - g_n) = 0$.

Define $\pi : L^\infty(G) \to \ell^\infty(\mathbb{N})$ by $\pi(f)(n) = m_n(f)$ for $f \in L^\infty(G)$ and $n \in \mathbb{N}$. Then it is clear that π is positive, $\pi(1) = 1$ and $\|\pi\| = 1$. For any $\ell \in \ell^\infty(\mathbb{N})$, let $f = \ell(1)g_1 + \ell(2)g_2 + \ldots$. Then $f \in L^\infty(G)$ and $\pi(f)(n) = m_n(f) = m_n\big(\ell(n)g_n\big) = \alpha\ell(n)$, i.e., $\pi(f) = \alpha\ell$, and $\|\pi(f)\|_\infty = \alpha \sup_{n\geq 1} |\ell(n)| = \alpha\|\ell\|_\infty$. Hence π is onto and, since $\pi(1) = 1$, $\|\pi^*\| = 1$. Thus, $\alpha\|\theta\| \leq \|\pi^*(\theta)\| \leq \|\theta\|$ for all $\theta \in \ell^\infty(\mathbb{N})^*$. It is clear that $\pi^*\mathcal{F} \subseteq LIM$ (see Miao [6], Theorem 2.5). We will show that $\pi^*\mathcal{F} \subseteq A_{g,\alpha}$. Let $\theta \in \mathcal{F}$. Then there exist a net $\{n_\beta\}$ in $\mathbb{N}$ such that $n_\beta \to \theta$ in the w^*-topology of $\ell^\infty(\mathbb{N})^*$. For any $h \in H$, $\pi^*\theta(h) = \theta(\pi h) = \lim_\beta (\pi h)(n_\beta) = \lim_\beta m_{n_\beta}(h) = 0$ and

$\pi^*\theta(g) = \theta(\pi g) = \lim_\beta (\pi g)(n_\beta) = \lim_\beta m_{n_\beta}(g) = \alpha$. So $\pi^*\theta \in \mathcal{A}_{g,\alpha}$. For θ_1 and θ_2 in $\mathcal{F}$ with $\theta_1 \neq \theta_2$, then $\|\theta_1 - \theta_2\| = 2$. Hence $\|\pi^*(\theta_1 - \theta_2)\| \geq \alpha\|\theta_1 - \theta_2\| = 2\alpha$. □

Corollary 1. *Let G be a σ-compact locally compact nondiscrete group. If G is amenable as a discrete group, then every nonempty G_δ-subset of LIM contains a norm discrete copy of $\beta\mathbb{N}$.*

Proof. Let $\mathcal{O}$ be a nonempty G_δ-subset of LIM. Then $\mathcal{O} = \bigcap_{k=1}^{\infty} \mathcal{O}_k$ for some open subsets $\mathcal{O}_k$ of LIM. Let $m_0 \in \mathcal{O}$. By the definition of the w^*-topology of LIM, $\mathcal{O}_k$ contains a neighborhood of m_0 in the form of $\{m \in LIM : \ \alpha_i \leq m(f_i) \leq \beta_i$ for $i = 1, 2, \ldots, n\}$, where α_i, β_i are numbers and $f_i \in L^\infty(G)$ for $i = 1, 2, \ldots, n$. For each k, we may assume, without loss of generality, that there are numbers a_i, b_i such that $\mathcal{O}_k = \{m \in LIM : \ a_i \leq m(h_i) \leq b_i$ for $i = n_k, n_k + 1, \ldots, n_{k+1}\}$ where $n_1 < n_2 < n_3 < \ldots$ is a sequence of natural numbers and $h_1, h_2, \ldots$ is a sequence of elements of $L^\infty(G)$. Let $f_i = h_i - m_0(h_i)$ for all i. Then the set $\{ m \in LIM : \ m(f_i) = 0$ for all $i \}$ contains m_0 and is a subset of $\mathcal{O}$. By our Main Theorem, it contains a norm discrete copy of $\beta\mathbb{N}$. □

Remark. This improves the main result of Miao [5]. Corollary 2 below is a "LIM" version of Granirer's theorem in [1] (see [7], (7.28)).

Corollary 2. *Let G be a σ-compact locally compact nondiscrete group. Let $\mathcal{C}$ be a compact, G_δ-subset of the spectrum of $L^\infty(G)$ and let $\{f_n\}$ be a sequence in $L^\infty(G)$. If*

$$\mathcal{A} = \{m \in LIM : supp \ \ m \subseteq \mathcal{C}, m(f_n) = 0 \text{ for all } n \geq 1\}$$

and G is amenable as a discrete group, then Exp $\mathcal{A} = \varnothing$. If $\mathcal{A} \neq \varnothing$, then $\mathcal{A}$ contains a norm discrete copy of $\beta\mathbb{N}$. Consequently, $\mathcal{A}$ is not separable in the norm topology.

Proof. Since $\mathcal{C}$ is a G_δ-set, we can find a sequence $\{\mathcal{C}_n\}$ of open sets of $\mathcal{S}$ such that $\bar{\mathcal{C}}_{n+1} \subset \mathcal{C}_n$ and $\mathcal{C} = \bigcap_{n=1}^{\infty} \mathcal{C}_n$. Since $\mathcal{S}$ is the maximal ideal space of $L^\infty(G)$, it is 0-dimensional. So we can suppose that each $\mathcal{C}_n$ is open and closed. Hence there are subsets U_n of G such that $\hat{U}_n = \mathcal{C}_n$ for all n, where $\hat{U}_n$ is defined by $\theta \in \hat{U}_n$ if and only if $\theta(1_{U_n}) = 1$. Thus,

$$\mathcal{A} = \{m \in LIM : \ m(f_n) = 0 \ \ and \ \ m(1_{G\sim U_n}) = 0 \ \text{ for all } \ n\}.$$

By our Main Theorm, Exp $\mathcal{A} = \varnothing$ and if $\mathcal{A} \neq \varnothing$, then $\mathcal{A}$ contains a norm discrete copy of $\beta\mathbb{N}$. □

Corollary 3. *Let G be a σ-compact locally compact nondiscrete group. Let Q be a closed G-invariant ideal of $L^\infty(G)$. If G is amenable as a discrete group and $\tilde{Q}$ is a G_δ-subset of $\mathcal{S}$, then every nonempty G_δ-subset of LIM_Q contains a norm discrete copy of $\beta\mathbb{N}$. In particular, LIM_Q has no exposed points.*

Proof. As in the proof of Corollary 2, since $\mathcal{C}$ is a G_δ-set, we can find a sequence $\{\mathcal{C}_n\}$ of open subsets of $\mathcal{S}$ such that $\bar{\mathcal{C}}_{n+1} \subset \mathcal{C}_n$ and $\mathcal{C} = \bigcap_{n=1}^{\infty} \mathcal{C}_n$; there are subsets U_n of G such that $\hat{U}_n = \mathcal{C}_n$ for all n, where $\hat{U}_n$ is defined by $\theta \in \hat{U}_n$ if and only if $\theta(1_{U_n}) = 1$. Thus, the set $\{m \in LIM : \ m(1_{G\sim U_n}) = 0 \ \text{ for all } \ n\}$ is equal to LIM_Q.

Let $\mathcal{O}$ be a nonempty G_δ-subset of LIM_Q. As in the proof of Corollary 1, we can assume that there is a sequence $\{f_n\}$ in $L^\infty(G)$ such that

$$\mathcal{O} = \{m \in LIM : m(f_n) = 0 \ \text{ for all } \ n\} \cap LIM_Q.$$

Hence $\mathcal{O} = \{m \in LIM : m(f_n) = 0 \ \text{ and } \ m(1_{G\sim U_n}) = 0 \ \text{ for all } \ n\}$ contains a norm discrete copy of βN by our Main Theorem. □

References

1. E. E. Granirer, *Exposed points of convex sets and weak sequential convergence*, Mem. Amer. Math. Soc. **123** (1972). MR **51:**1343
2. ______, *On finite equivalent invariant measures for semigroups of transformations*, Duke Math. J., **38** (1971), 395-408. MR **44:**404
3. ______, *Criteria for compactness and for discreteness of locally compact amenable groups*, Proc. Amer. Math. Soc. **40** (1973), 615-624. MR **49:**5712
4. F. P. Greenleaf, *Invariant Means on Topological Groups*, Van Nostrand, New York, 1969. MR **40:**4776
5. T. Miao, *The exposed points of the set of invariant means*, Trans. Amer. Math. Soc. **347** (1995), 1401-1408. MR **95g:**43003
6. ______, *On the sizes of the sets of invariant means*, Illinois J. Math. **36** (1992), 53-72. MR **93a:**43001
7. A. L. T. Paterson, *Amenability*, Amer. Math. Soc., Providence, Rhode Island, 1988. MR **90e:**43001
8. J. P. Pier, *Amenable Locally Compact Groups*, Wiley, New York, 1984. MR **90e:**43001
9. M. Talagrand, *Géométrie des simplexes de moyennes invariantes*, J. Funct. Anal. **34** (1979), 304-337. MR **80k:**43002
10. ______, *Invariant means on an ideal*, Trans. Amer. Math. Soc. **288** (1985), 257-272. MR **86e:**43005
11. ______, *Moyennes invariantes s'annulant sur des ideaux*, Compositio Math. **42** (1981), 213-216. MR **82b:**43003

Department of Mathematical Sciences, Lakehead University, Thunder Bay, Ontario, Canada P7E 5E1

E-mail address: tmiao@thunder.lakeheadu.ca