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THE ORDER OF A MERIDIAN OF A KNOTTED KLEIN BOTTLE

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Abstract. We consider the order of a meridian (of the group) of a Klein bottle smoothly embedded in the 4-sphere S^4. The order of a meridian of a Klein bottle in S^4 is a non-negative even integer. Conversely, we prove that, for every non-negative even integer n, there exists a Klein bottle in S^4 whose meridian has order n.

1. Introduction

Let F be a Klein bottle smoothly embedded in the 4-sphere S^4. Let $N(F)$ be the tubular neighborhood of F in S^4. Then $N(F)$ is a D^2-bundle over F and the boundary $\partial N(F)$ of $N(F)$ is an S^1-bundle over F. A fiber $S^1 \times \{*\}$, where $* \in F$, is called a *meridian* of F. Let G be the group of F, i.e., $\pi_1(S^4 - F)$. An element of G represented by a meridian of F is also called a *meridian* of G (or F). We consider the order of a meridian (of the group) of a Klein bottle F in S^4. Since $H_1(G) \cong Z_2$, the order of a meridian of a Klein bottle F in S^4 is a non-negative even integer. For example, the connected sum of any 2-knot and a standard Klein bottle in S^4 has a meridian of order 2. In this paper, more generally, we prove that

Theorem. *For every non-negative even integer n, there exists a Klein bottle in S^4 whose meridian has order n.*

2. Preliminaries

Let $K + h$ denote the Klein bottle in S^4 obtained by attaching a non-orientable 1-handle h to a 2-knot K in S^4. We may assume that the non-orientable 1-handle h on K is attached very near the base point of the group G $(= \pi_1(S^4 - K))$ of K. Then we can consider that the core of h represents an element $g(h)$ of G. Conversely, given an element g of G, there exists a non-orientable 1-handle h on K such that $g = g(h)$. For a subset S of a group H, we denote by $\langle\langle S\rangle\rangle$ the normal closure of S in H.

Proposition 1. *Let K be a 2-knot in S^4 and h a non-orientable 1-handle on K. Then the group of the Klein bottle $K + h$ in S^4 is given by $\pi_1(S^4 - K)/\langle\langle xg(h)xg(h)^{-1}\rangle\rangle$, where x is a meridian of K.*

Proof. In the same way as in Lemma 9 of [1], we can prove the proposition. □

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The following is immediately obtained from Proposition 1.

Lemma 2. *Let G_0 be a 2-knot group and x a meridian of G_0. Then, for any element g of G_0, the quotient group $G_0/\langle\langle xgxg^{-1}\rangle\rangle$ is the group of some Klein bottle in S^4.*

Lemma 3. *Let n be a non-negative even integer. Let H be a finitely generated group satisfying the following:*

(1) There exists an element h of H such that $\langle\langle h\rangle\rangle = H$.
(2) There exists an element u of H such that $uhu^{-1} = h^{-1}$.
(3) $h^{\frac{n}{2}} \in C(H)$, where $C(H)$ is the center of H.
(4) $|h| = n$, where $|h|$ denotes the order of h.

Then there exists a Klein bottle in S^4 whose meridian has order n.

Proof. Let $m = \frac{n}{2}$. By (1), there exist some 1-knot k and an epimorphism ϕ of $\pi_1(S^3 - k)$ onto H such that $\phi(x) = h$, where x is a meridian of k [2]. Let K be the m-twist-spun 2-knot of the 1-knot k. Then the group $\pi_1(S^4 - K)$ of K is given by $\pi_1(S^3 - k)/[x^m, \pi_1(S^3 - k)]$. Therefore, by (3), ϕ induces the epimorphism ϕ_* of $\pi_1(S^4 - K)$ onto H such that $\phi_*\alpha = \phi$, where $\alpha : \pi_1(S^3 - k) \to \pi_1(S^4 - K)$ is the natural homomorphism. Then $\alpha(x)$ is a meridian of K. Let $G = \pi_1(S^4 - K)/\langle\langle\alpha(x)\tilde{u}\alpha(x)\tilde{u}^{-1}\rangle\rangle$, where $\tilde{u} \in \phi_*^{-1}(u)$, and let $\beta : \pi_1(S^4 - K) \to G$ be the natural homomorphism. Then, by Lemma 2, G is the group of some Klein bottle in S^4 and $\beta\alpha(x)$ is a meridian of G. We show that $\beta\alpha(x)$ has order n in G. Since $\beta\alpha(x)^m \in C(G)$, we have

$$\beta\alpha(x)^m = \beta(\tilde{u})\beta\alpha(x)^m\beta(\tilde{u})^{-1} = \beta\alpha(x)^{-m}.$$

Therefore we see that $\beta\alpha(x)^{2m} = 1$. On the other hand, by (2), ϕ_* induces the epimorphism ψ of G onto H such that $\psi\beta\alpha = \phi_*\alpha = \phi$. Thus, since $\psi\beta\alpha(x) = \phi(x) = h$, it follows from (4) that $|\beta\alpha(x)| = |h| = 2m = n$. The proof is completed. □

Lemma 4. *Let n be a positive even integer. Let H_* be a finitely generated group satisfying the following:*

(a) There exists an element h_* of H_* such that $\langle\langle h_*\rangle\rangle = H_*$.
(b) There exists an element u_* of H_* such that $u_*h_*u_*^{-1} = h_*^{-1}$.
(c) $h_*^{\frac{n}{2}} \in C(H_*)$.
(d) $|h_*| = n$.

Then, for any positive integer m such that $(m, n) = 1$, there exists a Klein bottle in S^4 whose meridian has order mn.

Proof. Let $H = H_* \times A_{2m}$, where A_{2m} is the alternating group of degree $2m$. (Since m is odd, we have $2m = 2$ or > 5. Therefore A_{2m} is simple.) We will show that H satisfies the conditions (1)-(4) of Lemma 3.

(1) Let $h = h_*b$, where $b = (1, 2, ..., m)(m + 1, m + 2, ..., 2m) \in A_{2m}$. Then we have $|b| = m$ and $(|h_*|, |b|) = 1$. Therefore, by (a), we see that

$$H/\langle\langle h\rangle\rangle = H_* \times A_{2m}/\langle\langle h_*, b\rangle\rangle = A_{2m}/\langle\langle b\rangle\rangle \cong 1.$$

(2) Let $c = \{(1, m)(2, m - 1) \cdots (k, k + 2)\} \cdot \{(m + 1, 2m)(m + 2, 2m - 1) \cdots (3k + 1, 3k + 3)\} \in A_{2m}$, where $k = (m - 1)/2$. Then we have $cbc^{-1} = b^{-1}$. Let $u = u_*c$. Then, by (b), we get

$$uhu^{-1} = (u_*c)(h_*b)(c^{-1}u_*^{-1}) = (u_*h_*u_*^{-1})(cbc^{-1}) = h_*^{-1}b^{-1} = b^{-1}h_*^{-1} = h^{-1}.$$

(3) By (c), we have

$$h^{mn/2} = (h_* b)^{mn/2} = (h_*^{n/2})^m (b^{n/2})^m = (h_*^{n/2})^m \in C(H_*) \subset C(H).$$

(4) By (d), we get $|h| = |h_*| \cdot |b| = mn$.
Therefore, by Lemma 3, we complete the proof. □

The following corollary immediately follows from Lemma 4.

Corollary 5. *Let n be a positive even integer. If there exists a Klein bottle F in S^4 such that $x^{\frac{n}{2}} \in C(G)$ and $|x| = n$, where x is a meridian of the group G of F, then, for any positive integer m such that $(m, n) = 1$, there exists a Klein bottle in S^4 whose meridian has order mn.*

3. Proof of the Theorem

Case (1): $n = 0$.
Let

$$G_1 = \langle a, b, z : a = b^{-1}a^{-1}bab, zaz^{-1} = b^{-1}, zbz^{-1} = ab^{-1}a^{-1} \rangle$$

and $A = \langle a, b : a = b^{-1}a^{-1}bab \rangle$. The mapping of A to itself given by $a \to b^{-1}$ and $b \to ab^{-1}a^{-1}$ induces the automorphism of A. Therefore G_1 is the extension of A by the infinite cyclic group $\langle z : \rangle$. Since A is the group of the trefoil 1-knot, A is torsion-free and so is G_1. Let $y = aba^{-1}b^{-1}$. Then y commutes with z. Therefore the subgroup B_1 of G_1 generated by y and z is free abelian of rank 2. Let G_2 be the group of a non-trivial 1-knot and B_2 the peripheral subgroup of G_2. Let H be the free product of G_1 and G_2 with the subgroups B_1 and B_2 amalgamated under the isomorphism of B_1 on B_2 given by $y \to c$ and $z \to l$, where c and l are a meridian and a longitude of G_2. Then, if $h = b$ and $u = za$ in Lemma 3, we can see that H satisfies the conditions (1)-(4) of Lemma 3. Therefore there exists a Klein bottle in S^4 whose meridian has order 0.

Case (2): $n = 2(2k + 1)$ or $4(2k + 1), k \geq 0$.

First, when $n = 2(2k+1)$, let $H_* = Z_2$. Then, since the group of a standard Klein bottle in S^4 is Z_2, the assertion follows from Corollary 5. Next, when $n = 4(2k+1)$, let

$$(*) \qquad H_* = \langle a, b, x_* : a^2 = b^3 = (b^{-1}a)^3, x_* a x_*^{-1} = a^{-1}, x_* b x_*^{-1} = b^{-1}, x_*^2 = a^2 \rangle.$$

Then H_* is the extension of T_* by the cyclic group Z_2 of order 2, where T_* is the binary tetrahedral group presented by $\langle a, b : a^2 = b^3 = (b^{-1}a)^3 \rangle$. Then we can show that $\langle\langle x_* \rangle\rangle = H_*$, $a x_* a^{-1} = x_*^{-1}$, $x_*^2 \in C(H_*)$ and $|x_*| = |a| = 4$. Let $h_* = x_*$ and $u_* = a$ in Lemma 4. Then we can see that H_* satisfies the conditions (a)-(d) of Lemma 4. Therefore we complete the proof of case (2).

Case (3): $n = 8m$, $m > 0$.
Let

$$G = \langle x, y : y = x^m y^m x y^{-m} x^{-m}, x^{2m} y x^{-2m} = y^{-1} \rangle.$$

Then, by the second relation, we see that

$$x^{4m} y x^{-4m} = x^{2m} y^{-1} x^{-2m} = y.$$

Hence $y^{4m} = x^m y^m x^{4m} y^{-m} x^{-m} = x^{4m}$. Therefore we have

$$y^{-4m} = x^{2m} y^{4m} x^{-2m} = x^{2m} x^{4m} x^{-2m} = x^{4m} = y^{4m}.$$

It follows that $y^{8m} = 1$. Let $\phi : G \to SL(2,C)$ be the homomorphism defined by $\phi(x) = a$ and $\phi(y) = waw^{-1}$, where

$$a = \begin{pmatrix} e^{\pi i/4m} & 0 \\ 0 & e^{-\pi i/4m} \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} \frac{1}{2}(i+1) & \frac{1}{2}(1-i) \\ -\frac{1}{2}(i+1) & \frac{1}{2}(1-i) \end{pmatrix}.$$

Since $|a| = 8m$, we see that $|x| = |y| = 8m$.

Let $G_0 = \langle x, y : y = x^m y^m x y^{-m} x^{-m} \rangle$. Then G_0 is the group of a ribbon 2-knot. Therefore, by Lemma 2, G is the group of some Klein bottle in S^4.

Thus we complete the proof of the theorem.

4. Remarks

(1) The group G_0 of the 2-twist-spun 2-knot of the (-2,3,3)-pretzel 1-knot is presented by

$$\langle a, b, x_* : a^2 = b^3 = (b^{-1}a)^3, x_* a x_*^{-1} = a^{-1}, x_* b x_*^{-1} = b^{-1} \rangle.$$

Since $G_0/\langle\langle x_* a x_* a^{-1} \rangle\rangle$ is isomorphic to the group H_* given by (*), it follows from Lemma 2 that H_* is the group of some Klein bottle in S^4.

(2) The group G of a Klein bottle in S^4 has deficiency less than one because $H_1(G) \cong Z_2$. For each non-negative even integer n, the problem arises whether there exists a Klein bottle in S^4 whose group has a meridian of order n and has deficiency zero. From the proofs of cases (1) and (3) in Section 3, we can see that, if $n = 8m$ $(m \geq 0)$, then there exists such a Klein bottle in S^4. (In case of $n = 0$, for instance, the group presented by

$$\langle a, b, c, d, z : a = b^{-1}a^{-1}bab, zbz^{-1} = ab^{-1}a^{-1},$$
$$d = c^{-1}d^{-1}cdc,\ aba^{-1}b^{-1} = c, z = c^4 d^{-1} c^{-2} d^{-1} \rangle$$

is such an example because $\langle a, b, c, d, z : a = b^{-1}a^{-1}bab, d = c^{-1}d^{-1}cdc, aba^{-1}b^{-1} = c, z = c^4 d^{-1} c^{-2} d^{-1} \rangle$ is the group of a ribbon 2-knot.) The case $n \neq 8m$ $(m \geq 0)$ is still open.

(3) It is known that a meridian of a projective plane smoothly embedded in S^4 has order 2 or 4 (cf. [4, §VI]). In case of order 2, there exists such a projective plane in S^4 (e.g., the connected sum of any 2-knot and a standard projective plane in S^4 ([3], [4])). It remains open whether or not there exists a projective plane in S^4 whose meridian has order 4.

(4) Using the Theorem, we can easily prove that, for every non-negative even integer n, there exists a non-orientable surface of (non-orientable) genus $2m$ $(m > 0)$ in S^4 whose meridian has order n.

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