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A CHARACTERIZATION OF UNIFORM CONTINUITY FOR VOLTERRA EQUATIONS IN HILBERT SPACES

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Abstract. We show that the norm continuity of the resolvent for a Volterra equation of scalar type is equivalent to the decay to zero of a holomorphic operator family along some imaginary axis.

1. Introduction

Let X be a complex Banach space. In this work we consider the following Volterra equation of scalar type

$$u(t)=\int_0^t a(t-s)Au(s)ds+f(t), \quad t\in J:=[0,T], \tag{1.1}$$

where A is a closed linear, densely defined operator in X, and $a\in L^1_{loc}(\mathbb{R}_+)$ a scalar kernel. A continuous function $u:J\to X$ is called a strong solution of (1.1) on J if $u(t)\in D(A)$ for all $t\in J$, $Au(t)$ is continuous, and (1.1) holds on J. The basic concept concerning (1.1) is that of well-posedness which is the direct extension of the corresponding notion usually employed for the abstract Cauchy problem (of first order)

$$\dot{u}(t)=Au(t), \qquad u(0)=u_0. \tag{1.2}$$

In [4] it is shown that well-posedness is equivalent to the existence of a resolvent $\{S(t)\}_{t\geq 0}\subseteq \mathcal{B}(X)$ for (1.1), i.e. a strongly continuous family of bounded linear operators in X which commutes with A and satisfies the resolvent equation

$$S(t)x=x+\int_0^t a(t-s)AS(s)xds, \quad t\geq 0, \quad x\in D(A). \tag{1.3}$$

The resolvent is the central object to be studied in the theory of Volterra equations; it corresponds to the strongly continuous semigroup generated by A in the special case $a(t)=1$, i.e. for (1.2). The importance of the resolvent $S(t)$ is shown

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by the variation of parameters formula

$$u(t) = S(t)f(0) + \int_0^t S(t-s)\dot{f}(s)ds, \quad t \in J, \tag{1.4}$$

where $f \in W^{1,1}(J;X)$.

Due to the time invariance of (1.1), Laplace transform methods can be employed. Formally the Laplace transform $H(\lambda) = \hat{S}(\lambda)$ of the resolvent is represented by

$$H(\lambda) = (\lambda - \lambda\hat{a}(\lambda)A)^{-1}. \tag{1.5}$$

This formula leads to a characterization of Hille-Yosida type for existence of an at most exponentially growing resolvent for (1.1) in terms of properties of the holomorphic operator family $H(\lambda)$. For a general exposition of the theory, we refer to J. Prüss [4].

In this article we characterize the uniform continuity of the resolvent $S(t)$ for (1.1) on Hilbert spaces in terms of (1.5). We show that the norm continuity of $S(t)$ for $t > 0$ is equivalent to the decay to zero of $H(\lambda)$ along some imaginary axis. For the special case $a(t) = 1$ and A generator of a C_0-semigroup $T(t)$ in H, we have $S(t) = T(t)$ and the result reduces to a characterization of eventually norm continuous semigroups obtained recently by P. You [5] (see also [1] and [3]).

2. A CHARACTERIZATION IN HILBERT SPACES

Let H be a complex Hilbert space, A a closed linear unbounded operator in H with dense domain $D(A)$, and $a \in L^1_{loc}(\mathbb{R}_+)$ a scalar kernel $\neq 0$. We consider the Volterra equation

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)ds, \quad t \in J, \tag{2.1}$$

where $f \in C(J;H), J := [0,T]$.

Since (2.1) is a convolution equation on the halfline it is natural to employ the Laplace transform for its study. Besides the standing assumptions on $a(t)$ and A we therefore suppose that $a(t)$ is Laplace transformable, i.e. there is $\omega \in \mathbb{R}$ such that $\int_0^\infty e^{-\omega t}|a(t)|dt < \infty$. But we also have to restrict the class of resolvents.

We recall that $S(t)$ is called exponentially bounded of type (M, w), if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$||S(t)|| \leq Me^{\omega t}, \text{ for all } t \geq 0.$$

We will also need the following definition from [4].

Definition 2.1. Let $a \in L^1_{loc}(\mathbb{R}_+)$ be Laplace transformable and $k \in \mathbb{N}$. $a(t)$ is called k-regular if there is a constant $C > 0$ such that

$$|\lambda^n \hat{a}^{(n)}(\lambda)| \leq C|\hat{a}(\lambda)|$$

for all $Re\lambda > \omega$, $0 < n \leq k$.

Convolutions of k-regular kernels are again k-regular. Moreover, integration and differentiation are operations which preserve k-regularity as well. See [4], p. 70.

Our main result in this article is the following characterization.

Theorem 2.2. *Let A be a closed linear unbounded operator in a Hilbert space H with dense domain $D(A)$ and let $a \in L^1_{loc}(\mathbb{R}_+)$ satisfy $\int_0^\infty e^{-\omega t}|a(t)|dt < \infty$ for some $\omega \in \mathbb{R}$. Assume (2.1) admits a resolvent $S(t)$ of type (M, w), and $a(t)$ is 2-regular. Then the following conditions are equivalent.*

(a) $(S(t))_{t\geq 0}$ *is continuous in* $\mathcal{B}(H)$ *for* $t>0$,
(b) $\lim_{|\mu|\to\infty} ||H(\mu_0+i\mu)||=0$ *for some* $\mu_0>\omega$.

The above theorem reduces for the case $a(t)\equiv 1$ to the following result of P. You [5].

Corollary 2.3. *Let* A *be the generator of a strongly continuous semigroup* $T(t)$ *on a Hilbert space* H. *Then the following conditions are equivalent.*

(a) $(T(t))_{t\geq 0}$ *is norm continuous for* $t>0$,
(b) $\lim_{|\mu|\to\infty} ||R(\mu_1+i\mu;A)||=0$ *for some* $\mu_1>\omega$.

Uniform continuity of the resolvent $S(t)$ has a number of important consequences. For example, $u(t)$ defined by (1.4) in general is a solution of (1.1) if $Af\in W^{1,1}(J;H)$ or $f=a*g$ with $g\in W^{1,1}(J;H)$; if $S(t)$ is $\mathcal{B}(H)$-continuous, then $Af\in BV(J;H)$ or $f=a*g$ with $g\in BV(J;H)$ are already sufficient.

Another type of application concerns stability of (1.1) or, in other terms, integrability of $S(t)$ (see [4], section 10).

For example, suppose $b\in L^1_{loc}(\mathbb{R}_+)$ satisfies $\int_0^\infty |b(t)|e^{-\beta t}dt<\infty$ for some $\beta\geq 0$, and assume there is a resolvent $S_b(t)$ for

$$v(t)=g(t)+\int_0^t b(t-s)Av(s)ds,\quad t\in\mathbb{R}_+,$$

such that $\omega_b=\overline{\lim}_{t\to\infty}t^{-1}log|S_b(t)|<\infty$.

Let $c(t)$ be a completely positive function with associated creep function

$$k(t)=\kappa+\omega t+\int_0^t k_1(s)ds,\quad t>0,$$

where $\kappa,\omega\geq 0, k_1(t)\geq 0$ nonincreasing with $\lim_{t\to\infty}k_1(t)=0$, and let $\alpha=\omega+k_1(0^+)$. According to [4], Theorem 4.1, there is a resolvent $S_a(t)$ for (1.1), where $a\in L^1_{loc}(\mathbb{R}_+)$ is defined by $\hat{a}(\lambda)=\hat{b}(1/\hat{c}(\lambda))$, for $Re\lambda$ sufficiently large.

As a direct consequence of Theorem 2.2 and [4], Theorem 10.4, we obtain the following result.

Theorem 2.4. *Let* H *be a Hilbert space. Suppose the function*

$$g(z,\omega)=w^{-1}(\hat{b}(z)/\hat{b}(z+z\omega)-1)$$

is analytic at $(\infty,0)$, *let* $1/\kappa+\alpha<\infty$ *and assume* $\alpha>\omega_b$, *and* $b(t)$ *is 2-regular. Then* $S_a(t)$ *is uniformly integrable if* $S_a(t)$ *is integrable and*

$$\lim_{|\mu|\to\infty}||H_b(\mu_0+i\mu)||=0$$

for some sufficiently large μ_0.

In order to prove Theorem 2.2 we will need the following lemmas.

Lemma 2.5. *Let* $a\in L^1_{loc}(\mathbb{R}_+)$ *be Laplace transformable and assume that*

$$H(\lambda)=(\lambda-\lambda\hat{a}(\lambda)A)^{-1}$$

exists for all $Re\lambda>\omega$. *Then there are functions* $k_1(\lambda),k_2(\lambda)$ *and* $h_1(\lambda),h_2(\lambda)$, $h_3(\lambda)$ *such that*

(i) $H'(\lambda)=k_1(\lambda)H(\lambda)+k_2(\lambda)H(\lambda)^2,\ Re\lambda>\omega$,
(ii) $H''(\lambda)=h_1(\lambda)H(\lambda)+h_2(\lambda)H(\lambda)^2+h_3(\lambda)H(\lambda)^3,\ Re\lambda>\omega$.

Proof. First, we observe that $\hat{a}(\lambda) \neq 0$ for $Re\lambda > \omega$ because A is unbounded ([4], p. 43). Next, let $H(\lambda) = p(\lambda)(q(\lambda) - A)^{-1}$ where $p(\lambda) := \frac{1}{\lambda \hat{a}(\lambda)}$ and $q(\lambda) := \frac{1}{\hat{a}(\lambda)}$. Then,

$$H'(\lambda) = \frac{p'(\lambda)}{p(\lambda)} H(\lambda) - \frac{q'(\lambda)}{p(\lambda)} H(\lambda)^2, \quad Re\lambda > \omega,$$

and

$$H''(\lambda) = \frac{p''(\lambda)}{p(\lambda)} H(\lambda) + [\frac{2q'(\lambda)p'(\lambda)}{p(\lambda)^2} - \frac{q''(\lambda)}{p(\lambda)}] H(\lambda)^2 - \frac{2q'(\lambda)^2}{p(\lambda)^2} H(\lambda)^3, \quad Re\lambda > \omega.$$

□

Remark. A calculation shows us that

$$k_1(\lambda) := -(\frac{1}{\lambda} + \frac{\hat{a}'(\lambda)}{\hat{a}(\lambda)}), \qquad k_2(\lambda) := \frac{\lambda \hat{a}'(\lambda)}{\hat{a}(\lambda)}, \qquad Re\lambda > \omega, \tag{2.2}$$

and

$$h_1(\lambda) := \frac{2\hat{a}'(\lambda)}{\lambda \hat{a}(\lambda)} + \frac{2}{\lambda^2} + \frac{2\hat{a}'(\lambda)^2}{\hat{a}(\lambda)^2} - \frac{\hat{a}''(\lambda)}{\hat{a}(\lambda)}, \qquad Re\lambda > \omega, \tag{2.3}$$

$$h_2(\lambda) := \frac{2\hat{a}'(\lambda)}{\hat{a}(\lambda)} + \frac{\lambda \hat{a}''(\lambda)}{\hat{a}(\lambda)}, \; h_3(\lambda) := \frac{-2\lambda^2 \hat{a}'(\lambda)^2}{\hat{a}(\lambda)^2}, \; Re\lambda > \omega. \tag{2.4}$$

As a direct consequence of formulae (2.2)-(2.4) we obtain the following:

Lemma 2.6. *Let $a \in L^1_{loc}(\mathbb{R}_+)$ be Laplace transformable and 2-regular, then there exists a constant $M > 0$ such that*

(i) $|\lambda^{1-j} k_{1+j}(\lambda)| < M$ *for all* $Re\lambda > \omega$; $\quad j = 0, 1,$

(ii) $\int_{|\mu| \geq N} |h_j(\mu_0 + i\mu)|^j d\mu < M$ *for all* $\mu_0 > \omega$ *and* $N > 1; j = 1, 2,$

(iii) $Sup_{|\mu| \geq N} |h_3(\mu_0 + i\mu)| < M$ *for all* $\mu_0 > \omega$ *and* $N > 1$.

We will need also the following result from [1].

Lemma 2.7. *Let X be a Banach space and let $R : [0, \infty) \to X$ be a function which is continuous for $t > 0$. Moreover, assume that there exist $M > 0, \omega \in \mathbb{R}$ such that $||R(t)|| \leq Me^{\omega t}$. Then the Laplace transform $\hat{R}(\lambda) := \int_0^\infty e^{-\lambda t} R(t) dt$ exists for all $\lambda \in \mathbb{C}$ satisfying $Re\lambda > \omega$ and*

$$\lim_{|\mu| \to \infty} \hat{R}(\mu_0 + i\mu) = 0$$

for every $\mu_0 > \omega$.

We can now prove our main result. The proof is inspired by the proof of O. El Mennaoui and K.-J. Engel [1] for the semigroup case.

Proof of Theorem 2.2. (a) implies (b). We use Lemma 2.7 choosing $X = \mathcal{B}(H)$ and $R(t) = S(t)$.

(b) implies (a). Let $x \in H$ be fixed. Because $||S(t)e^{-\mu_0 t} x|| \leq Me^{\omega_0 t} ||x||$ with $\omega_0 := \omega - \mu_0 < 0$, the function $t \to \chi_{[0,\infty)}(t) e^{-\mu_0 t} S(t) x$ is in $L^2(\mathbb{R}; H)$, where $\chi_{[0,\infty)}(t)$ denotes the characteristic.

Since H is a Hilbert space, by Plancherel's theorem the Fourier transform is a unitary operator on $L^2(\mathbb{R};H)$. Thus we obtain

$$\mathcal{F}(\chi_{[0,\infty)}(\cdot)e^{-\mu_0\cdot}S(\cdot)) = H(\mu_0+i\mu)x.$$

Hence,

$$S(t)e^{-\mu_0 t}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\mu t}H(\mu_0+i\mu)x d\mu, \tag{2.5}$$

for $t>0$ and each $x \in H$. Clearly, the resolvent $S(t)$ is continuous in $\mathcal{B}(H)$ for $t>0$ if and only if $S_{\mu_0}(t) := S(t)e^{-\mu_0 t}$ is continuous in $\mathcal{B}(H)$ for $t>0$. Next, note that for $x \in D(A)$ we have

$$H(\mu_0+i\mu)x = \hat{a}(\mu_0+i\mu)H(\mu_0+i\mu)Ax + \frac{1}{\mu_0+i\mu}x.$$

Observe that $\hat{a}(\mu_0+i\mu) \to 0$ as $|\mu| \to \infty$ by the Riemann-Lebesgue lemma. In particular $\lim_{|\mu|\to\infty} H(\mu_0+i\mu)x = 0$. Using this and integrating by parts in (2.5) we obtain

$$S_{\mu_0}(t)x = \frac{-1}{2\pi t}\int_{-\infty}^{\infty} e^{i\mu t}H'(\mu_0+i\mu)x d\mu$$

for all $x \in D(A), t>0$. Next, by Lemma 2.5(i) and Lemma 2.6(i) we also get $\lim_{|\mu|\to\infty} H'(\mu_0+i\mu)x = 0$. Using this and again integration by parts we obtain the formula

$$S_{\mu_0}(t)x = \frac{1}{\pi t^2}\int_{-\infty}^{\infty} e^{i\mu t}H''(\mu_0+i\mu)x d\mu \tag{2.6}$$

for all $x \in D(A), t>0$.

Next we show that the operator family $(t^2 S_{\mu_0}(t))$ is continuous in $\mathcal{B}(H)$ for $t>0$. In fact, formula (2.6) shows

$$\begin{aligned}||t^2 S_{\mu_0}(t)x - s^2 S_{\mu_0}(s)x|| &= \frac{1}{\pi}||\int_{-\infty}^{\infty}(e^{i\mu t}-e^{i\mu s})H''(\mu_0+i\mu)x d\mu|| \\ &\le \frac{1}{\pi}||\int_{|\mu|\ge N}(e^{i\mu t}-e^{i\mu s})H''(\mu_0+i\mu)x d\mu|| \\ &\quad + \frac{1}{\pi}\int_{|\mu|\le N}|e^{i\mu t}-e^{i\mu s}|\,||H''(\mu_0+i\mu)x||d\mu \\ &=: I_1(N)+I_2(N).\end{aligned}$$

Let $\epsilon > 0$. We show that $I_1(N) < \epsilon||x||$ for N sufficiently large. To this end take $x^* \in H$ and observe that by Lemma 2.5(ii) we have

$$\begin{aligned}
&\frac{1}{2}|\langle\int_{|\mu|\geq N}(e^{i\mu t}-e^{i\mu s})H''(\mu_0+i\mu)xd\mu, x^*\rangle| \\
&\quad\leq \int_{|\mu|\geq N}|\langle h_1(\mu_0+i\mu)H(\mu_0+i\mu)x,x^*\rangle|d\mu \\
&\qquad+\int_{|\mu|\geq N}|\langle h_2(\mu_0+i\mu)H(\mu_0+i\mu)^2x,x^*\rangle|d\mu \\
&\qquad+\int_{|\mu|\geq N}|\langle h_3(\mu_0+i\mu)H(\mu_0+i\mu)^3x,x^*\rangle|d\mu \\
&\quad\leq Sup_{|\mu|\geq N}||H(\mu_0+i\mu)||\Bigg((\int_{|\mu|\geq N}|h_1(\mu_0+i\mu)|d\mu)||x||\,||x^*|| \\
&\qquad+(\int_{|\mu|\geq N}||H(\mu_0+i\mu)x||^2d\mu)^{1/2}(\int_{|\mu|\geq N}||h_2(\mu_0+i\mu)x^*||^2d\mu)^{1/2} \\
&\qquad+Sup_{|\mu|\geq N}|h_3(\mu_0+i\mu)|(\int_{|\mu|\geq N}||H(\mu_0+i\mu)x||^2d\mu)^{1/2} \\
&\qquad\qquad\times(\int_{|\mu|\geq N}||H(\mu_0+i\mu)^*x^*||^2d\mu)^{1/2}\Bigg)
\end{aligned}$$

where we used the Cauchy-Schwartz and Hölder inequalities. Next, by the Plancherel theorem for the Hilbert space valued Fourier transform (see [2], Lemma 2) and Lemma 2.6(ii) and (iii) , we get the estimate

$$\begin{aligned}
\leq\quad & Sup_{|\mu|\geq N}||H(\mu_0+i\mu)||\Bigg(M||x||\,||x^*||+(2\pi M\int_0^\infty||S_{\mu_0}(t)x||^2dt)^{1/2}||x^*|| \\
+\quad & M(2\pi\int_0^\infty||S_{\mu_0}(t)x||^2dt)^{1/2}(2\pi\int_0^\infty||S_{\mu_0}(t)^*x^*||^2dt)^{1/2}\Bigg).
\end{aligned}$$

Since $(S_{\mu_0}(t))_{t\geq0}$ and the adjoint $(S_{\mu_0}(t)^*)_{t\geq0}$ are exponentially bounded of type (M,ω_0) for some $\omega_0<0$, there exists a constant $C>0$ such that

$$2\pi\int_0^\infty||S_{\mu_0}(t)x||^2dt\leq C^2||x|| \quad\text{and}\quad 2\pi\int_0^\infty||S_{\mu_0}(t)^*x^*||^2dt\leq C^2||x^*||.$$

Combining this with the above estimate we obtain a constant $K>0$ such that

$$\begin{aligned}
I_1(N) &= Sup_{||x^*||\leq1}|\langle\int_{|\mu|\geq N}(e^{i\mu t}-e^{i\mu s})H''(\mu_0+i\mu)xd\mu,x^*\rangle| \\
&\leq 2KSup_{|\mu|\geq N}||H(\mu_0+i\mu)||\,||x||.
\end{aligned}$$

Since $\lim_{|\mu|\to\infty}||H(\mu_0+i\mu)||=0$ there exists $N>0$ such that

$$2KSup_{|\mu|\geq N}||H(\mu_0+i\mu)||<\epsilon$$

which yields the desired estimate $I_1(N)<\epsilon||x||$ for every $x\in D(A)$.

In order to estimate $I_2(N)$ we observe that $|e^{-i\alpha} - 1|^2 = 4\sin^2(\alpha/2), \alpha \in \mathbb{R}$. Therefore, for the above fixed N we have

$$\begin{aligned}
&\int_{|\mu|\le N} |e^{-i\mu(s-t)} - 1|\, ||H''(\mu_0 + i\mu)x|| d\mu \\
&\quad \le 2\int_{|\mu|\le N} |\sin(\frac{(s-t)\mu}{2})|\, ||H''(\mu_0 + i\mu)x|| d\mu \\
&\quad \le |s-t| N \int_{|\mu|\le N} ||H''(\mu_0 + i\mu)x|| d\mu.
\end{aligned}$$

Using these estimates for $I_1(N), I_2(N)$ and the fact that $D(A)$ is dense in H yields $||t^2 S_{\mu_0}(t) - s^2 S_{\mu_0}(s)|| < 2\epsilon$ for all $|s-t| < \delta$. This completes the proof. $\square$

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