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A MAXIMUM PRINCIPLE FOR P-HARMONIC MAPS WITH L^q FINITE ENERGY

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Abstract. We show a maximum principle for P-harmonic maps with L^q-finite energy. As an application we can generalize a non-existence theorem for harmonic maps with finite Dirichlet integral by Schoen and Yau to those maps.

1. Introduction

Let $f : (M, g) \to (N, h)$ be a smooth map of Riemannian manifolds. We are interested in determining under which condition the boundedness of the length $|df|$ of df can be induced. This problem is related to the maximum principle for solutions of an elliptic operator of second order in a sense. For instance several Schwarz type lemmas for harmonic maps, which state the boundedness of $|df|$ under a certain negative curvature condition of the target manifold (N, h), are known and a few Liouville type theorems are induced from them (cf. [C-Y], [E-L], [G-H], [Sh], [Y-2]). As a degenerate case of the negativity condition of the curvature of (N, h) the following observation is interesting in view of our aspect (see also [C]).

Theorem*. *Let $f : (M, g) \to (N, h)$ be a smooth map from a non-compact complete (connected) Riemannian manifold (M, g) of dimension m into a Riemannian manifold (N, h). Let Γ_M (resp. Γ_N) denote the set of points of M (resp. N) where Ric_M (resp. Riem_N) is not non-negative (resp. not non-positive). Let Σ and E be subsets of M such that $\Sigma := \Gamma_M \cup f^{-1}(\Gamma_N) \cup E$ and $M \setminus \overline{\Sigma} \neq \varnothing$. Suppose $f : (M \setminus \overline{\Sigma}, g) \to (N, h)$ is harmonic, i.e. $\mathrm{Trace}_g \nabla(df) = 0$ on $M \setminus \overline{\Sigma}$, and $\int_{M\setminus\overline{\Sigma}} |df|^q < +\infty$ for $q > 1$. Then the following holds:*

$$|df|(x) \le \sup_{y\in\Sigma} |df|(y) \quad \text{for any } x \in M,$$

where the right hand side of the above inequality may take infinity and is defined to be zero if $\Sigma = \varnothing$.

To show this result the subharmonicity of $|df|$ on $M \setminus \overline{\Sigma}$ plays an essential role in view of the Weitzenböck formula. In fact Theorem* can be obtained by [Sc-Y], Theorem 1, and applying the following general statement to $|df|$ (see also [L-S]).

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Proposition. *Let* (M, g) *be as above and let* v *be a smooth function on* M *satisfying* $\Delta v := \mathrm{Trace}_g \nabla^2 v \geq 0$ *outside a proper subset* A *of* M. *Suppose the interior of* $\{v \leq \sup_A v\}$ *is not empty and* $\int_{M \setminus A} |dv^q| < +\infty$ *with* $q \geq 1$. *Then* $v(x) \leq \sup_A v$ *for any* $x \in M$.

For the proof the reader should see [Y-1], Corollary and the proof of Theorem 3 (see also §2). Here the L^1-integrability condition of dv^q is essential and cannot be removed.

For a positive number $p > 1$ $f : (M, g) \to (N, h)$ is said to be *p-harmonic* if f satisfies the following:

$$\mathrm{Trace}_g \nabla(|df|^{p-2} df) = 0 \quad \text{on} \quad M.$$

As in the case of a harmonic map, i.e. $p = 2$, any p-harmonic map $f : (M, g) \to (N, h)$ can be characterized as a critical point of the L^p energy functional $\int_M |df|^p$ if it is finite. In this note we generalize Theorem* to p-harmonic maps. However we cannot apply the above proposition to the case $p \neq 2$ because the subharmonicity of $|df|^2$ breaks down. Nevertheless we will show the following by modifying an idea developed in [T].

Theorem. *Let* $f : (M, g) \to (N, h)$ *and* Σ *be as in Theorem**. *Suppose* $f : (M \setminus \overline{\Sigma}, g) \to (N, h)$ *is p-harmonic, i.e.* $\mathrm{Trace}_g \nabla(|df|^{p-2} df) = 0$ *on* $M \setminus \overline{\Sigma}$, *and* $\int_{M \setminus \overline{\Sigma}} |df|^q < +\infty$ *for* $q > p - 1$ *with* $p \geq 2$. *Then the following holds:*

$$|df|(x) \ \leq \sup_{y \in \Sigma} |df|(y) \quad \textit{for any } x \in M.$$

This result can be regarded as a maximum principle for L^q integrable solutions of a non-linear degenerate elliptic equation. In particular, we obtain the following (cf. [L-S], [N], [Sc-Y]).

Corollary. *Let* (M, g) *and* (N, h) *be a non-compact complete* (*connected*) *Riemannian manifold with non-negative Ricci curvature and a Riemannian manifold with non-positive Riemannian curvature respectively. Then any p-harmonic map* $f : (M, g) \to (N, h)$ *with* L^q *finite energy is constant if* $q > p - 1$ *with* $p \geq 2$ ($q \geq 1$ *when* $p = 2$).

2. Proof of the Theorem

By the Weitzenböck formula (cf. [E-L], (2.20) and (3.13)) we obtain the following:

$$\frac{1}{2} \Delta |df|^2 = |\nabla(df)|^2 - \langle df, (DD^* + D^* D)(df) \rangle + Q(df),$$

where D^* is the adjoint operator of

$$D : \Gamma(\bigwedge^{s} TM^* \otimes f^* TN) \to \Gamma(\bigwedge^{s+1} TM^* \otimes f^* TN)$$

and $Q(df)$ is given by the following:

$$Q(df) := \sum_{i=1}^{m} \langle \mathrm{Ric}_M(df(e_i)), df(e_i) \rangle - \sum_{j,k=1}^{m} \langle \mathrm{Riem}_N(df(e_j), df(e_k)) df(e_k), df(e_j) \rangle.$$

Since $Q(df) \geq 0$ on $M \setminus \overline{\varSigma}$ by the curvature condition and $D(df) = 0$, we obtain for any real number q

$$\begin{aligned} &\frac{1}{2}|df|^{q-2}\varDelta|df|^2 + |df|^{q-2}\langle df, DD^*(df)\rangle \\ &\qquad \geq |df|^{q-2}|\nabla(df)|^2 \quad \text{on} \quad M \setminus (\overline{\varSigma} \cup \{|df| = 0\}). \end{aligned} \tag{1}$$

If $c_0 := \sup_{y \in \varSigma} |df|(y) = +\infty$, then the assertion is trivial. We have only to consider the case $0 \leq c_0 < +\infty$. Let $M = M_* \cup M_+$, $M_* := \{|df| \leq c_0\}$, and $M_+ := \{|df| > c_0\}$. Assume $M_+ \neq \varnothing$. First we take a smooth non-negative function λ on a real line satisfying $\lambda \equiv 0$ on $(-\infty, c_0^2]$, $\lambda > 0, \lambda' \geq 0$ on $(c_0^2, +\infty)$ and $\lambda \equiv 1$ on $[c_1, +\infty)$, $c_1 > c_0^2$. By the completeness of g, for a fixed point $x_* \in M$ and any $r > 0$ there exists a non-negative Lefschetz continuous function ω on M such that $0 \leq \omega \leq 1$ on M, $\omega \equiv 1$ on $B_r(x_*)$, $\omega \equiv 0$ on $M \setminus B_{2r}(x_*)$, and $|d\omega| \leq C/r$ on M for $C > 0$ not depending on r, where $B_r(x_*)$ is the geodesic ball centered at $x_* \in M$ of radius $r > 0$. Setting $u := |df|$ it should be noted that $\overline{\varSigma} \cup \{u = 0\} \subset M_*$, u is smooth on M_+, and $\lambda(u(x)^2) > 0$ if and only if $x \in M_+$. By (1) we have

$$\begin{aligned} \int \lambda(u^2)\omega^2 u^{q-2}|\nabla(df)|^2 \leq \frac{1}{2}&\int \lambda(u^2)\omega^2 u^{q-2}\varDelta u^2 \\ &+ \int \lambda(u^2)\omega^2 u^{q-2}\langle df, DD^*(df)\rangle. \end{aligned} \tag{2}$$

Using integration by parts we have

$$\begin{aligned} \int \lambda(u^2)\omega^2 u^{q-2}\varDelta u^2 &= -\int \langle \nabla(\lambda(u^2)\omega^2 u^{q-2}), \nabla u^2\rangle \\ &= -\int \lambda'(u^2)\omega^2 u^{q-2}|\nabla u^2|^2 - 2\int \lambda(u^2)\omega u^{q-2}\langle \nabla\omega, \nabla u^2\rangle \\ &\quad - \int \lambda(u^2)\omega^2\langle \nabla u^{q-2}, \nabla u^2\rangle \\ &\leq \frac{2}{\varepsilon}\int \lambda(u^2)\omega^2 u^q|\nabla\omega|^2 - 2(q-2-\varepsilon)\int \lambda(u^2)\omega^2 u^{q-2}|\nabla u|^2 \end{aligned} \tag{$*$}$$

for any $q > 0$ and $\varepsilon > 0$. Here we remark the following:

$$D^*(vu^{p-2}df) = vD^*(u^{p-2}df) - u^{p-2}\mathbf{e}(dv)^*df$$

on $\{u > 0\}$ for any real number p and any smooth function v on M, where $\mathbf{e}(dv)^*$ is the adjoint operator of the left exterior multiplication by dv. In particular, by the p-harmonicity of f, i.e. $D^*(u^{p-2}df) = 0$, on M_+ we obtain

$$D^*(vu^{q-2}df) = -u^{p-2}\mathbf{e}(d(vu^{q-p}))^*df, \tag{3}$$

$$D^*(df) = (p-2)\mathbf{e}(d\log u)^*df \tag{4}$$

on M_+ for any $p > 1$ and $q > 0$. Using integration by parts, and the Cauchy-Schwarz inequalities $|\mathbf{e}(du)^* df| \le |\nabla u| u$ and $|D^*(df)| \le \sqrt{m}|\nabla(df)|$, we obtain

$$\begin{aligned}
\int \lambda(u^2)\omega^2 u^{q-2}\langle df, DD^*(df)\rangle &= \int \langle D^*(\lambda(u^2)\omega^2 u^{q-2} df), D^*(df)\rangle \\
&= -\int \Big\{2\lambda'(u^2)u^{q-1} + (q-p)u^{q-3}\lambda(u^2)\Big\}\omega^2\langle \mathbf{e}(du)^* df, D^*(df)\rangle \\
&\quad - 2\int \lambda(u^2)\omega u^{q-2}\langle \mathbf{e}(d\omega)^* df, D^*(df)\rangle \quad \text{by} \quad (3) \\
(**) \qquad &\le -\min\{0, (q-p)(p-2)\}\int \lambda(u^2)u^{q-2}\omega^2|\nabla u|^2 \\
&\quad + 2\int \lambda(u^2)\omega u^{q-1}|\nabla\omega|\,|D^*(df)| \text{ by (4)} \\
&\le -\min\{0, (q-p)(p-2)\}\int \lambda(u^2)u^{q-2}\omega^2|\nabla u|^2 \\
&\quad + \varepsilon\int \lambda(u^2)\omega^2 u^{q-2}|\nabla(df)|^2 + \frac{m}{\varepsilon}\int \lambda(u^2)u^q|\nabla\omega|^2
\end{aligned}$$

for any $q > 0$ and $\varepsilon > 0$. Since $|\nabla u| \le |\nabla(df)|$ on $\{u > 0\}$, by (2), $(*)$, and $(**)$, we obtain for any $q > 0$ and $0 < \varepsilon \le 1$

$$\begin{aligned}
\big(q - 1 + \min\{0, (q-p)(p-2)\} - 2\varepsilon\big)&\int \lambda(u^2)\omega^2 u^{q-2}|\nabla u|^2 \\
&\le \frac{m+1}{\varepsilon}\int \lambda(u^2)u^q|\nabla\omega|^2.
\end{aligned}$$

If $q > p-1$ with $p \ge 2$, then $\varepsilon := \min\{1, q - 1 + \min\{0, (q-p)(p-2)\}\}/3 > 0$. Since $\lambda' \ge 0$ and $\sup\lambda' = 1$, we obtain

$$\int_{B_r(x_*)} \lambda(u^2)u^{q-2}|\nabla u|^2 \quad \le \quad \frac{(m+1)C^2}{\varepsilon^2 r^2}\int_{M\setminus\overline{\Sigma}} u^q.$$

By the L^q integrability condition of u on $M \setminus \overline{\Sigma}$ and letting r tend to infinity we obtain

$$\int_M \lambda(u^2)u^{q-2}|\nabla u|^2 = 0.$$

Since $\lambda \ge 0$ and $\lambda(u(x)^2) > 0$ if and only if $x \in M_+$, the above equality implies $\nabla u \equiv 0$ on M_+, i.e. $u = |df| \equiv \text{constant} > c_0$ on M_+ by the connectedness of M. If $\Sigma \neq \varnothing$, then by continuity there exists a point $x_* \in M_*$ such that $|df|(x_*) > c_0$, which is a contradiction and so M_+ is empty. If $\Sigma = \varnothing$, i.e. $c_0 = 0$, then $u \equiv c > 0$ on M. Hence by [Y-1], Theorem 7, $\int_M |df|^q = c^q \times \mathrm{Vol}(M) = +\infty$, which is also a contradiction and so M_+ is empty, i.e. f is constant. This completes the proof of Theorem. In case of $p = 2$ and $q = 1$, the assertion of Corollary follows from [L-S],Theorem 2.2,(b).

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