

A NOTE ON TOEPLITZ OPERATORS ON DISCRETE GROUPS

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ABSTRACT. We study Toeplitz algebras associated to partially-ordered and quasi-partially ordered discrete groups.

INTRODUCTION

In the past thirty years, the classical Toeplitz operators have been extended to various contexts. One important extension is to replace (Z, Z_+) by some general ordered (or even just partially ordered) discrete abelian group (G, G_+) and study the corresponding Toeplitz operators; for related materials, see [1]–[5]. In recent years, there has been much progress in studying Toeplitz operators on ordered groups. By comparison, less is known about Toeplitz operators on general partially ordered groups. To analyze such Toeplitz operators, a universal Toeplitz algebra is constructed in [1], and this plays a key role in the subsequent work of G. Murphy. It is proved in [1] that when (G, G_+) is an ordered group, the universal Toeplitz algebra is isomorphic to the Toeplitz algebra defined in the usual way. However, it is shown in [7] that the analogous statement is not true for Toeplitz algebras on general partially ordered groups.

To study Toeplitz algebras on general partially ordered groups, one approach is the following: given a partially ordered group, find suitable ordered groups that contain the partially ordered group, and then use the Toeplitz algebras on the ordered groups to understand the Toeplitz algebra on the partially ordered group. This approach was first taken by E. Park in [5], and was generalized by the authors in [7] and [8]. To do this, it is necessary to extend partially ordered groups to a more general setting; in [7], quasi-ordered groups are introduced. In this paper, we extend this notion to the nonabelian case in order to study Toeplitz operators on discrete nonabelian groups.

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1. THE NATURAL MORPHISM BETWEEN TOEPLITZ ALGEBRAS

Let G be a discrete group, $\{\delta_g|g \in G\}$ be the usual orthonormal basis for $\ell^2(G)$, where

$$\delta_g(h) = \begin{cases} 1, & \text{if } g = h, \\ 0, & \text{otherwise} \end{cases}$$

for $g, h \in G$. For any $g \in G$, we denote by L_g and R_g the unitary operators on $\ell^2(G)$ defined by

$$\begin{aligned} L_g(\delta_h) &= \delta_{gh}, \\ R_g(\delta_h) &= \delta_{hg^{-1}}, \end{aligned}$$

for all $h \in G$.

The von Neumann algebra generated by $\{L_g|g \in G\}$ is called the group von Neumann algebra of G and is denoted by $W^*(G)$. The C^* -algebra generated by $\{L_g|g \in G\}$ is called the reduced C^* -algebra of G and is denoted by $C_r^*(G)$. It is well-known that $W^*(G)$ is the commutant of the set $\{R_g|g \in G\}$.

For any $E \subseteq G$, let $\ell^2(E)$ be the closed subspace of $\ell^2(G)$ generated by $\{\delta_g|g \in E\}$; its projection is denoted by p^E . For any $T \in W^*(G)$, we define the compression $T^{(E)}$ of T to $\ell^2(E)$ by letting

$$T^{(E)} = p^E T |_{\ell^2(E)}.$$

We refer to $T^{(E)}$ as a Toeplitz operator, and call T the symbol of $T^{(E)}$. We also define generalised ‘‘unilateral shifts’’ $T_g, g \in G$, by letting

$$T_g = p^E L_g |_{\ell^2(E)} \quad (g \in G).$$

Definition. The C^* -algebra generated by $\{p^E L_g p^E | g \in E\}$ is denoted by $\mathcal{T}^E(G)$, and is called the Toeplitz algebra with respect to E . Its commutator ideal is denoted by $C(\mathcal{T}^E(G))$.

Let $G_+ \subseteq G$, we say that (G, G_+) is a quasi-partially ordered group (below we briefly denote it by qpo), if $e \in G_+, G_+ \cdot G_+ \subseteq G_+$, and $G = G_+ \cdot G_+^{-1}$, where e is the unit of G and $G_+^{-1} = \{g^{-1}|g \in G_+\}$. Furthermore if $G = G_+ \cup G_+^{-1}$, then (G, G_+) is referred as a quasi-ordered group. Note that when $G_+^0 = G_+ \cap G_+^{-1} = \{e\}$, then qpo groups are just the usual partially ordered groups.

Natural question. Suppose that (G, G_1) and (G, G_2) are two qpo groups, can the morphism γ^{G_2, G_1} defined by $\gamma^{G_2, G_1}(p^{G_1} L_g p^{G_1}) = p^{G_2} L_g p^{G_2}$ for any $g \in G$ be extended as a C^* -algebra morphism from $\mathcal{T}^{G_1}(G)$ to $\mathcal{T}^{G_2}(G)$?

Note that if it is true, then for any $g_1 \in G_1$,

$$\begin{aligned} \gamma^{G_2, G_1}(p^{G_1} L_{g_1^{-1}} p^{G_1} L_{g_1} p^{G_1}) &= \gamma^{G_2, G_1}(1) = \gamma^{G_2, G_1}(p^{G_1} L_e p^{G_1}), \\ \text{so } p^{G_2} L_{g_1^{-1}} p^{G_2} L_{g_1} p^{G_2} &= p^{G_2} L_e p^{G_2} = 1. \end{aligned}$$

In particular, $(p^{G_2} L_{g_1^{-1}} p^{G_2} L_{g_1} p^{G_2})\delta_e = \delta_e$, so $g_1 \in G_2$. Since g_1 is arbitrary, we know that $G_1 \subseteq G_2$. So in the following we always assume $G_1 \subseteq G_2$.

Definition. Let E_1 and E_2 be two subsets of G with $E_1 \subseteq E_2$, we say that E_2 is finitely decomposed by E_1 if for any finite subset F of G , there exists an element $g \in G$, such that for all $s \in F$, the following hold:

- (a) $sg \in E_2$ iff $s \in E_2$.
- (b) $sg \in E_1 \cup E_2^C$.

Theorem 1.1. *Let E_1 and E_2 be two subsets of G with $E_1 \subseteq E_2$, if E_2 is finitely decomposed by E_1 , then γ^{E_2, E_1} is well-defined and can be extended as a C^* -algebra morphism from $\mathcal{T}^{E_1}(G)$ to $\mathcal{T}^{E_2}(G)$.*

Proof. Let T be an operator in $\mathcal{T}^{E_1}(G)$ of the form

$$T = \sum_{i=1}^m \xi_i \prod_{j=1}^{n_i} p^{E_1} L_{g_{ij}} p^{E_1};$$

then

$$\gamma^{E_2, E_1}(T) = \sum_{i=1}^m \xi_i \prod_{j=1}^{n_i} p^{E_2} L_{g_{ij}} p^{E_2} \quad \text{for } \xi_i \in C, g_{ij} \in G.$$

To show that γ^{E_2, E_1} is well-defined and can be extended as a C^* -algebra morphism, it suffices to show that $\|\gamma^{E_2, E_1}(T)\| \leq \|T\|$, since such kind of T is dense in $\mathcal{T}^{E_1}(G)$.

$\forall \varepsilon > 0$, there exists some $f \in \ell^2(E_2)$ with finite support such that $\|f\| = 1$, and $\|\gamma^{E_2, E_1}(T)\| \leq \|\gamma^{E_2, E_1}(T)f\| + \varepsilon$.

Let $f = \sum_{p=1}^n \eta_p \delta_{h_p}$, set $F = \{h_p, (\prod_{j=m_i}^{n_i} g_{ij})h_p | p = 1, 2, \dots, n, i = 1, 2, \dots, m, 1 \leq m_i \leq n_i\}$. By assumption, there exists an element $g \in G$, such that for all $s \in F$, we have

$$(1.1) \quad sg \in E_2 \quad \text{iff} \quad s \in E_2,$$

$$(1.2) \quad sg \in E_1 \cup E_2^C.$$

By (1.1), we know that

$$R_{g^{-1}} \gamma^{E_2, E_1}(T)f = \gamma^{E_2, E_1}(T) R_{g^{-1}} f.$$

By (1.2), we have

$$\begin{aligned} \gamma^{E_2, E_1}(T) R_{g^{-1}} f &= \left(\sum_{i=1}^m \xi_i \prod_{j=1}^{n_i} p^{E_2} L_{g_{ij}} p^{E_2} \right) R_{g^{-1}} f \\ &= \left(\sum_{i=1}^m \xi_i \prod_{j=1}^{n_i} p^{E_2} p^{E_1 \cup E_2^C} L_{g_{ij}} p^{E_2} p^{E_1 \cup E_2^C} \right) R_{g^{-1}} f = TR_{g^{-1}} f \end{aligned}$$

so

$$\begin{aligned} \|\gamma^{E_2, E_1}(T)\| &\leq \|\gamma^{E_2, E_1}(T)f\| + \varepsilon \\ &= \|R_{g^{-1}} \gamma^{E_2, E_1}(T)f\| + \varepsilon = \|TR_{g^{-1}} f\| + \varepsilon \leq \|T\| + \varepsilon. \end{aligned}$$

□

Remark. When E_2 is finitely decomposed by E_1 and $e \in E_2$, then for any finite subset F of G , we can further choose $g \in E_1$ satisfying the condition. In fact, replacing F by $F \cup \{e\}$ will do. So for qpo groups (G, G_1) and (G, G_2) with $G_1 \subseteq G_2$, G_2 is finitely decomposed by G_1 , iff for any finite subset $F = F_1 \cup F_2$ with $F_1 \subseteq G_2 \setminus G_1$ and $F_2 \subseteq G_2^C$, there exists $g \in G_1$ such that $F_1 g \subseteq G_1$ and $F_2 g \subseteq G_2^C$.

Definition. For any subset E of G , we say E is finitely liftable, if G is finitely decomposed by E .

Examples. (1) Let (G, G_+) be a qpo group, then G_+ is finitely liftable. In fact, it is easy to show inductively that for any finite subset F of G , there is a g in G_+ such that $Fg \subseteq G_+$.

(2) Recall for any subset S of G , S is said to be almost invariant, if $S \setminus gS$ is finite for all $g \in G$. By Lemma 4.1 in [6], we know that if S is almost invariant, then S is finitely liftable.

Proposition 1.2. For any subset E of G , if E is finitely liftable, then we have the following exact sequence:

$$0 \longrightarrow \text{Ker } \gamma^{G,E} \longrightarrow \mathcal{T}^E(G) \xrightarrow{\gamma^{G,E}} C_r^*(G) \longrightarrow 0$$

with the property that

(1) $\text{Ker } \gamma^{G,E} \subseteq K(\ell^2(E))$ if and only if E is almost invariant, where $K(\ell^2(E))$ is the ideal of compact operators on $\ell^2(E)$.

(2) If (G, E) is a ppo group, then $\text{Ker } \gamma^{G,E} \subseteq C(\mathcal{T}^E(G))$ and $\text{Ker } \gamma^{G,E} = C(\mathcal{T}^E(G))$ if and only if G is abelian. In this case, we have the following exact sequence:

$$0 \longrightarrow C(\mathcal{T}^E(G)) \longrightarrow \mathcal{T}^E(G) \xrightarrow{\sigma^E} C(\widehat{G}) \longrightarrow 0$$

where \widehat{G} is the Pontryagin dual group of G , $\sigma^E(p^E L_g p^E) = \varepsilon_g$ for $g \in G$, and $\varepsilon_g(\gamma) = \gamma(g)$ for $\gamma \in \widehat{G}$.

Proof. By Theorem 1.1 we know that $\gamma^{G,E}$ is well-defined and can be extended as a C^* -algebra morphism, so we have the following exact sequence:

$$0 \longrightarrow \text{Ker } \gamma^{G,E} \longrightarrow \mathcal{T}^E(G) \xrightarrow{\gamma^{G,E}} C_r^*(G) \longrightarrow 0.$$

Define a bounded linear operator ρ^E from $C_r^*(G)$ to $B(\ell^2(E))$ by $\rho^E(T) = p^E T p^E$ for $T \in C_r^*(G)$, then the C^* -algebra generated by $\rho^E(C_r^*(G))$ equals $\mathcal{T}^E(G)$ and $\gamma^{G,E} \circ \rho^E = 1$, so $\text{Ker } \gamma^{G,E} = \{T - \rho^E \circ \gamma^{G,E}(T) | T \in \mathcal{T}^E(G)\}$.

(1) Since $\text{Ker } \gamma^{G,E} = \{T - \rho^E \circ \gamma^{G,E}(T) | T \in \mathcal{T}^E(G)\}$, it easily follows that $\text{Ker } \gamma^{G,E} \subseteq K(\ell^2(E))$ if and only if E is almost invariant.

(2) If (G, E) is a qpo group, let $T = \prod_{i=1}^n p^E L_{g_i} p^E, g_i \in G$ for $i = 1, 2, \dots, n$, then $T - \rho^E \circ \gamma^{G,E}(T) = \prod_{i=1}^n p^E L_{g_i} p^E - p^E L_{g_1 g_2 \dots g_n} p^E$. Let $F = \{g_n, g_{n-1} g_n, \dots, g_1 g_2 \dots g_n\}$, since E is finitely liftable, there is a g in E , such that $Fg \subseteq E$. Let $S = 1 - p^E L_g p^E L_{g^{-1}} p^E$, then $S \in C(\mathcal{T}^E(G))$ and it is easy to show that $(T - \rho^E \circ \gamma^{G,E}(T))S = T - \rho^E \circ \gamma^{G,E}(T)$, which implies that $T - \rho^E \circ \gamma^{G,E}(T)$ belongs to $C(\mathcal{T}^E(G))$. Since the linear span of such kind of T is dense in $\mathcal{T}^E(G)$, by the continuity of ρ^E and $\gamma^{G,E}$, we know that $\text{Ker } \gamma^{G,E} \subseteq C(\mathcal{T}^E(G))$. If $\text{Ker } \gamma^{G,E} = C(\mathcal{T}^E(G))$, then for any $g_1, g_2 \in G$, $T = p^E L_{g_1} p^E L_{g_2} p^E - p^E L_{g_2} p^E L_{g_1} p^E \in C(\mathcal{T}^E(G))$, so $\gamma^{G,E}(T) = 0$, which implies $g_1 g_2 = g_2 g_1$, so G is abelian. On the otherhand, if G is abelian, then $C_r^*(G)$ is abelian, and since $\mathcal{T}^E(G)/\text{Ker } \gamma^{G,E}$ is isomorphic to an abelian C^* -algebra, we know that in this case $C(\mathcal{T}^E(G)) \subseteq \text{Ker } \gamma^{G,E}$, so $\text{Ker } \gamma^{G,E} = C(\mathcal{T}^E(G))$.

When G is abelian, it is amenable which implies that the regular representation of $C^*(G) \cong C(\widehat{G})$ (\widehat{G} is compact since G is discrete and abelian) is faithful. For any $g \in G$, the regular representation of δ_g is L_g , and its Fourier transform is ε_g , thus the conclusion holds. □

Next we show that the condition of finitely decomposable stated in Theorem 1.1 in some sense is also necessary.

Let (G, G_1) and (G, G_2) be two qpo groups with $G_1 \subseteq G_2$, define a linear operator $\rho : \mathcal{T}^{G_2}(G) \rightarrow B(\ell^2(G_1))$ by $\rho(T) = p^{G_1} T p^{G_1}$, $T \in \mathcal{T}^{G_2}(G)$. The C^* -subalgebra of $B(\ell^2(G_1))$ generated by $\rho(\mathcal{T}^{G_2}(G))$ is denoted by $\mathcal{R}^{G_1}(G)$. Clearly $\mathcal{R}^{G_1}(G)$ contains $\mathcal{T}^{G_1}(G)$ and a dense subalgebra of $\mathcal{R}^{G_1}(G)$ consists of operators of the form

$$T = \sum_{l=1}^N \xi_l \left(\prod_{i=1}^{n_l} p^{G_1} \left(\prod_{j=1}^{k_i} p_{l_{ij}} L_{g_{l_{ij}}} \right) p^{G_1} \right)$$

where each $p_{l_{ij}}$ equals p^{G_1} or p^{G_2} .

Proposition 1.3. *Let (G, G_1) and (G, G_2) be two qpo groups with $G_1 \subseteq G_2$, then the natural morphism γ defined by $\gamma(T) = \sum_{l=1}^N \xi_l (\prod_{i=1}^{n_l} p^{G_2} (\prod_{j=1}^{k_i} p^{G_2} L_{g_{l_{ij}}}) p^{G_2})$ can be extended as a C^* -algebra morphism from $\mathcal{R}^{G_1}(G)$ to $\mathcal{T}^{G_2}(G)$, if and only if G_2 is finitely decomposed by G_1 .*

Proof. If γ is well-defined and can be extended as a C^* -algebra morphism, we show that in this case G_2 is finitely decomposed by G_1 . If not, then there exists a finite subset $F = F_1 \cup F_2$ with $F_1 \subseteq G_2 \setminus G_1$ and $F_2 \subseteq G_2^C$, such that for any $g \in G_1$, either $(F_1 g) \cap G_1^C \neq \emptyset$ or $(F_2 g) \cap G_2 \neq \emptyset$. Since $e \in G_1, G_1 \cdot G_1 \subseteq G_1$, and $G = G_1 \cdot G_1^{-1}$, it is easy to show that both F_1 and F_2 are non-empty. Let $F_1 = \{g_1, g_2, \dots, g_n\}$ and $F_2 = \{h_1, h_2, \dots, h_l\}$.

For any $g \in G$, define $T_g^{(i)} = p^{G_i} L_{g^{-1}} p^{G_i} L_g p^{G_i}$ for $i = 1, 2$; set

$$T = T_{g_1}^{(1)} T_{g_2}^{(1)} \dots T_{g_n}^{(1)} (1 - T_{h_1}^{(2)}) (1 - T_{h_2}^{(2)}) \dots (1 - T_{h_l}^{(2)}) p^{G_1} \in \mathcal{R}^{G_1}(G).$$

Then it is easy to show that $T = 0$, so $\gamma(T) = 0$.

On the other hand, $\gamma(T) = T_{g_1}^{(2)} T_{g_2}^{(2)} \dots T_{g_n}^{(2)} (1 - T_{h_1}^{(2)}) (1 - T_{h_2}^{(2)}) \dots (1 - T_{h_l}^{(2)}) p^{G_2}$ and $\gamma(T) \delta_e = \delta_e \neq 0$, which is a contradiction.

For the sufficient part, see the proof of Theorem 1.1. □

Proposition 1.4. *Let (G, G_1) and (G, G_2) be two qpo groups with $G_1 \subseteq G_2$, if (G, G_1) is quasi-ordered, then G_2 is finitely decomposed by G_1 .*

Proof. For any finite subset F of G , let $F = \{g_1, g_2, \dots, g_n, h_1, h_2, \dots, h_l\}$ with $\{g_1, g_2, \dots, g_n\} \subseteq G_2$ and $\{h_1, h_2, \dots, h_l\} \subseteq G_2^C$. Let $g_1, g_2, \dots, g_k \in G_1$, and $g_{k+1}, g_{k+2}, \dots, g_n \in G_2 \setminus G_1$, then $g_{k+1}, g_{k+2}, \dots, g_n \in G_2 \cap G_1^{-1} \subseteq G_2^0$. Since $G_2^C \cdot G_2^0 = G_2^C$, it remains only to prove that for any finite subset $\{l_1, l_2, \dots, l_m\} \subseteq G_2 \cap G_1^{-1}$, there is a $g \in G_2^0 \cap G_1$ such that $l_i g \in G_1$ for all i . When $m = 1, l_1 \in G_2 \cap G_1^{-1}$, set $g = l_1^{-1}$ will do. If for $\{l_1, l_2, \dots, l_{m-1}\}$, there is a $g' \in G_2^0 \cap G_1$, such that $l_1 g', l_2 g', \dots, l_{m-1} g' \in G_1$, in this case, if $l_m g' \in G_1$, then set $g = g'$; otherwise $l_m g' \in G_1^{-1} \cap G_2^0$, so $(l_m g')^{-1} \in G_1 \cap G_2^0$. Let $g = g' (l_m g')^{-1}$, then $l_i g \in G_1$ for all i . □

Proposition 1.5. *If $G \neq \{e\}$, let (G, G_1) and (G, G_2) be two ordered groups, define $G_+^2 = G_1 \times G_2, G_{\ell ex}^2 = \{(g_1, g_2) \mid g_1 \in G_1 \setminus \{e\} \text{ or } g_1 = e \text{ and } g_2 \in G_2\}$, then $\gamma_{G_{\ell ex}^2, G_+^2}$ cannot be extended as a C^* -algebra morphism.*

Proof. Let $E_1 = G_1 \times G$ and $E_2 = G \times G_2$, then $E_1 \cap E_2 = G_+^2$. If $\gamma^{G_{\ell ex}^2, G_+^2}$ is well-defined and can be extended, then choose $g_1 \in G_1 \setminus \{e\}$, and $g_2 \in G_2 \setminus \{e\}$, let $T = p^{E_1} p^{E_2} L_{(g_1, g_2)} p^{E_2} L_{(g_1^{-1}, g_2^{-1})} p^{E_1} p^{E_2}$, we have

$$\begin{aligned} T &= p^{E_1} p^{E_2} L_{(g_1, g_2)} L_{(g_1^{-1}, e)} p^{E_2} L_{(g_1, e)} L_{(g_1^{-1}, g_2^{-1})} p^{E_1} p^{E_2} \\ &= p^{E_1} p^{E_2} L_{(e, g_2)} p^{E_2} L_{(e, g_2^{-1})} p^{E_1} p^{E_2} \\ &= p^{E_1} p^{E_2} L_{(e, g_2)} p^{E_1} p^{E_2} L_{(e, g_2^{-1})} p^{E_1} p^{E_2}, \end{aligned}$$

so $T \in \mathcal{T}^{G_+^2}(G^2)$, and

$$\gamma^{G_{\ell ex}^2, G_+^2}(T) = p^{G_{\ell ex}^2} L_{(e, g_2)} p^{G_{\ell ex}^2} L_{(e, g_2^{-1})} p^{G_{\ell ex}^2}.$$

On the other hand,

$$\begin{aligned} T &= p^{E_1} p^{E_2} L_{(g_1, g_2)} L_{(g_1^{-2}, e)} p^{E_2} L_{(g_1^2, e)} L_{(g_1^{-1}, g_2^{-1})} p^{E_1} p^{E_2} \\ &= p^{G_+^2} L_{(g_1^{-1}, g_2)} p^{G_+^2} L_{(g_1, g_2^{-1})} p^{G_+^2}, \end{aligned}$$

thus we have

$$p^{G_{\ell ex}^2} L_{(g_1^{-1}, g_2)} p^{G_{\ell ex}^2} L_{(g_1, g_2^{-1})} p^{G_{\ell ex}^2} = 1.$$

So $\gamma^{G_{\ell ex}^2, G_+^2}(T) = 1$, which is a contradiction since $(e, g_2^{-1}) \notin G_{\ell ex}^2$. □

2. TOEPLITZ OPERATORS ON DIRECTED SETS

Let E be a subset of G , in this section, we always assume the following two conditions hold:

- (1) E is a directed set, the partial order on it is denoted by \leq .
- (2) For any finite subset F of G , there is a $g_0 \in E$ such that $Fg \subseteq E$ for all $g \geq g_0$.

Examples. (1) Let (G, E) be a partially ordered group, for any $g_1, g_2 \in G$, we say that $g_1 \leq g_2$ if $g_1^{-1}g_2 \in E$. Since for any $g_1, g_2 \in E$, there is a $g \in E$ such that $g_1^{-1}g \in E$ and $g_2^{-1}g \in E$, E is a directed set and satisfies condition (2) above.

(2) Let E be a countably infinite subset of G , if E is almost invariant, then E will satisfy the conditions.

By definition, we know that E is finitely liftable, so we have the following exact sequence:

$$0 \longrightarrow Ker \gamma^{G, E} \longrightarrow \mathcal{T}^E(G) \longrightarrow C_r^*(G) \longrightarrow 0.$$

So for any $T \in C_r^*(G)$, we have $\|T\| = \|\rho^E(T)\| = \|T^{(E)}\|$, it still holds for all $T \in W^*(G)$, since it easily shows that $\lim_{g \in E} R_g p^E R_g^* = 1$ in the strong operator topology on $B(\ell^2(G))$. Replace $K(\ell^2(E))$ or $C(\mathcal{T}^E(G))$ by general $Ker \gamma^{G, E}$, a unified method to deal with Toeplitz operators on directed sets can be given. For details, see [6] and Section 3 of [1].

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