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# A NOTE ON INVARIANCE OF SPECTRUM FOR SYMMETRIC BANACH $^*$-ALGEBRAS

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ABSTRACT. Let $A$ be a symmetric Banach $^*$-algebra, let $B$ be a Banach algebra, and assume that $A \subseteq B$. A result is proved giving conditions which imply that every element of $A$ has the same spectrum in both $A$ and $B$.

## INTRODUCTION

Let $A$ be a Banach algebra. The complete algebra norm on $A$ will be denoted $\| \|_A$. For $a \in A$, $\sigma_A(a)$ is the spectrum of $a$ in $A$. Now assume that, in addition, $A$ is a $^*$-algebra. Then $A$ is symmetric if $\sigma_A(a^*a) \subseteq [0, \infty)$ for all $a \in A$. Basic information concerning symmetric $^*$-algebras can be found in [4, Chapter IV, Section 7]. All $C^*$-algebras are symmetric [4, Theorem (4.6.9)]. Also, Shirali's Theorem [1, Theorem 5, p. 226] implies that a Banach $^*$-algebra $A$ is symmetric if and only if $A$ is hermitian; meaning, $\sigma_A(a) \subseteq \mathbf{R}$ for all $a = a^* \in A$.

There has been recent interest in the question:

If $A$ is a $C^*$-algebra, and $\pi\colon\ A \to B(X)$ is an isomorphism of $A$ into the algebra of all bounded linear operators on a Banach space $X$, then when does it hold that $\sigma_A(a) = \sigma(\pi(a))$, the operator spectrum of $\pi(a)$, for all $a \in A$?

Some results on this question can be found in [3]. Here we prove a theorem concerning symmetric $^*$-algebras which has some bearing on this question. A corollary of this theorem generalizes some results in [3].

## THE RESULTS

**Theorem.** *Let $A$ and $B$ be Banach algebras with $A$ a* closed *subalgebra of $B$. When $A$ has a unit, then assume that this element is also the unit of $B$. Further, assume that $A$ is a symmetric $^*$-algebra with continuous involution such that either*

(i) *the embedding of $A$ in $B$ is continuous*; *or*
(ii) *$A$ is semsimple.*

*Then $\sigma_A(a) = \sigma_B(a)$ for all $a \in A$.*

*Proof.* First note that in either case (i) or (ii), the norms of $A$ and $B$ are equivalent on $A$. For if (i) holds, then this follows from the Open Mapping Theorem, and when (ii) holds, this is a consequence of Johnson's Uniqueness of Norm Theorem.

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Now assume that $B$ has a unit 1, and $1 \in A$. Assume $a \in A$ with $a^{-1} \in B$. By symmetry, $(n^{-1} + aa^*)^{-1} \in A$ for all integers $n \geq 1$. Suppose $aa^*$ is not invertible in $A$. Then by a standard argument,

$$b_n = (n^{-1} + aa^*)^{-1} / \|(n^{-1} + aa^*)^{-1}\|_A$$

has the properties $b_n = b_n^* \in A$, $\|b_n\|_A = 1$, and $\|aa^* b_n\|_A \to 0$ as $n \to \infty$. Since the norms on $A$ and $B$ are equivalent on $A$, and

$$\|a^* b_n\|_B = \|a^{-1}(aa^* b_n)\|_B \leq \|a^{-1}\|_B \|aa^* b_n\|_B \to 0,$$

we have $\|a^* b_n\|_A \to 0$. Thus, taking the involution, $\|b_n a\|_A \to 0$, so $\|b_n a\|_B \to 0$. Multiplying $\{b_n a\}$ on the right by $a^{-1}$ in $B$, we have $\|b_n\|_B \to 0$, and finally $\|b_n\|_A \to 0$, a contradiction. The contradiction implies that $aa^*$ is invertible in $A$. Set $c = (aa^*)^{-1} \in A$, so $aa^* c = 1$, and therefore, $a^{-1} = a^* c \in A$.

In the case where $A$ does not have a unit, one can be adjoined in the usual way (adjoin the unit of $B$ if $B$ has a unit). The resulting unital $^*$-algebra $A_1$ is symmetric and a closed subalgebra of $B_1$ (or $B$). The result then follows from the argument above. □

The corollary we now prove applies to the question in the Introduction.

**Corollary 1.** *Let $A$ be a $C^*$-algebra. Assume $\pi\colon A \to B$ is a continuous algebra homomorphism of $A$ into a Banach algebra $B$. Set $J = \ker(\pi)$. If $e$ is a unit for $A$ modulo $J$, then assume $\pi(e)$ is the unit for $B$. For all $a \in A$*

$$\sigma_{A/J}(a + J) = \sigma_B(\pi(a)).$$

*Proof.* By standard results which hold for $C^*$-algebras, $J$ is a closed $^*$-ideal of $A$, $A/J$ is a $C^*$-algebra, and $A/J$ is symmetric. Also, by a result of S. Cleveland [2, Lemma 5.3], the image $\pi(A)$ is a closed subalgebra of $B$. Therefore the Theorem applies to prove the result. □

The next corollary generalizes some results in [3].

**Corollary 2.** *Let $A$ be a $C^*$-algebra which is a subalgebra of a Banach algebra $D$. Assume that there exists a constant $M > 0$ such that $\|a\|_D \leq M\|a\|_A$ for all $a \in A$. Let $K$ be a closed subspace of $D$ with the property that for all $a \in A$ and $k \in K$, $ak \in K$. Let $\pi\colon A \to B(K)$ be defined by $\pi(a)k = ak$, $a \in A$, $k \in K$. Set $J = \ker(\pi)$. If $e$ is a unit for $A$ modulo $J$, then assume $ek = k$ for all $k \in K$.*

*Then $\sigma_{A/J}(a + J) = \sigma_{B(K)}(\pi(a))$ for all $a \in A$.*

*Proof.* For all $a \in A$ and $k \in K$,

$$\|\pi(a)k\|_D = \|ak\|_D \leq \|a\|_D \|k\|_D \leq M\|a\|_A \|k\|_D.$$

Therefore, $a \to \pi(a)$ is a continuous homomorphism of $A$ into $B(K)$. Thus, Corollary 1 implies the result. □

There are some interesting applications of the Theorem to the representation theory of $L^1(G)$. These will be considered in a subsequent paper which is joint work of the present author with Professor Ajit I. Singh.

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