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STABILITY OF THE WULFF SHAPE

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Abstract. We consider the functional of a hypersurface, given by a convex elliptic integrand with a volume constraint. We show that, up to homothety and translation, the only closed, oriented, stable critical point is the Wulff shape.

1. Introduction

Let $F : S^n \longrightarrow \mathbf{R}^+$ be a smooth function and denote its gradient with respect to the standard metric on S^n by DF. Consider the map

$$\phi : S^n \longrightarrow \mathbf{R}^{n+1},$$

$$\nu \longrightarrow F\nu + DF,$$

and consider the (possibly singular) hypersurface defined by $W_F := \phi(S^n)$. The differential of ϕ is given by

$$d\phi_\nu = (D^2F + F1)_\nu, \tag{1}$$

where D^2F denotes the intrinsic Hessian of F on S^n and 1 denotes the identity on $T_\nu S^n$. If we impose the condition that

$$(D^2F + F1)_\nu > 0, \quad \forall \nu \in S^n, \tag{2}$$

where > 0 means that the matrix is positive definite, then W_F is a smooth, convex hypersurface in $\mathbf{R}^{n+1}$ called the *Wulff shape* of F. We assume from now on that this is the case.

Now let

$$X : \Sigma \longrightarrow \mathbf{R}^{n+1}$$

be a smooth immersion of a compact, oriented hypersurface, possibly with nonempty boundary. Let

$$\nu : \Sigma \longrightarrow S^n$$

denote its Gauss map. We assign to each such X the functional

$$J_F(X) := \int_\Sigma F(\nu) dA_X. \tag{3}$$

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Such functionals have been treated by many authors (see for example [1], page 517, and the references therein). The algebraic $(n+1)$-volume enclosed by Σ is given by

$$V(X) = \frac{1}{n+1}\int_\Sigma \langle X, \nu\rangle dA_X.$$

We are interested in those hypersurfaces which are critical points of the functional J_F restricted to those hypersurfaces enclosing a fixed volume V (and spanning a fixed boundary in the case that $\partial\Sigma \neq \emptyset$). By a standard argument involving Lagrange multipliers, this means we are considering critical points of the functional

$$J_{F,\Lambda}(X) := \int_\Sigma F(\nu) dA_X + \Lambda V(X), \tag{4}$$

where Λ is a constant. We will show below that such critica are characterized by the Euler-Lagrange equation

$$div_\Sigma DF - nhF + \Lambda = 0 \tag{5}$$

where h is the mean curvature of Σ. In (5) the constant $\Lambda = \Lambda_\Sigma$ can be determined for a critical immersion.

We will call a critical immersion X *stable* if and only if the second variation of J_F (or equivalently of $J_{F,\Lambda}$) is non-negative for all (compactly supported) variations of X preserving the enclosed $(n+1)$-volume. We have the following result.

Theorem 1. *Let* $X : \Sigma \longrightarrow \mathbf{R}^{n+1}$ *be a smooth immersion of an oriented, closed* ($\partial\Sigma = \emptyset$)*, stable critical point of* $J_{F,\Lambda}$*. Then, up to translation,* $X(\Sigma) = tW_F$*, where* $t = -n/\Lambda_\Sigma$ *.*

A result known as Wulff's Theorem states that W_F actually *minimizes* J_F among all hypersurfaces having the same volume, so the stability of W_F is obvious. Proofs of Wulff's Theorem can be found in [3] and [4].

Note that the case $F \equiv 1$ is the widely studied class of constant mean curvature hypersurfaces. For this special case Theorem 1 was first obtained by Barbosa and do Carmo [2]. Later, Wente [5] gave a closely related but more direct proof which avoided the use of the Jacobi operator. The proof of Theorem 1 is a modification of Wente's idea to the case of non-constant F.

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2. Preliminaries

Let $X : \Sigma \longrightarrow \mathbf{R}^{n+1}$ be a smooth oriented hypersurface with Gauss map $\nu : \Sigma \longrightarrow S^n$. Consider a variation of X of the form

$$X_t = X + t(\psi\nu + \xi) + O(t^2)$$

where ψ is a smooth function on Σ and ξ is tangent to X. (Both ψ and ξ are assumed to have compact support if $\partial\Sigma \neq \emptyset$.) The corresponding first variation of the normal is given by

$$\delta\nu = -\nabla\psi + d\nu(\xi)$$

and the first variation of the volume element is

$$\delta dA = (div\xi - nh\psi)dA.$$

We use this to compute the first variation of J_F,

$$\begin{aligned}\delta J_F(X) &= \int_\Sigma (\langle DF(\nu), \delta\nu\rangle + F(\nu)\delta dA)\\ &= \int_\Sigma (\langle DF(\nu), -\nabla\psi\rangle - nhF\psi dA)\\ &\quad + \int_\Sigma (\langle DF(\nu), d\nu(\xi)\rangle + F div\xi)dA.\end{aligned}$$

Note that $T_{\nu(p)}S^n = dX(T_p\Sigma)$, so that if we denote the inner product on this plane by $\langle\ ,\ \rangle$, then

$$\langle(\nabla F\circ\nu), \cdot\rangle = d(F\circ\nu) = \nu^* dF = \langle DF(\nu), d\nu(\cdot)\rangle = \langle d\nu(DF), \cdot\rangle. \tag{6}$$

From this it follows that

$$\nabla(F\circ\nu) = d\nu(DF) \tag{7}$$

Integrating by parts, we then obtain

$$\begin{aligned}\delta J_F(X) &= \int_\Sigma (\psi(divDF - nhF))dA + \int_\Sigma (\langle d\nu(DF), \xi\rangle + F div\xi)dA\\ &= \int_\Sigma (\psi(divDF - nhF))dA + \int_\Sigma (\langle\nabla F), \xi\rangle + F div\xi)dA\\ &= \int_\Sigma (\psi(divDF - nhF))dA + \int_\Sigma (-F div\xi + F div\xi)dA\\ &= \int_\Sigma (\psi(divDF - nhF))dA.\end{aligned}$$

Finally, we recall that the first variation of the volume $V(X)$ is

$$\delta V(X) = \int_\Sigma \psi dA$$

Hence we obtain the Euler -Lagrange equation for the functional $J_{F,\Lambda}$:

$$divDF - nhF + \Lambda = 0. \tag{8}$$

Under the convexity condition (2) this equation is elliptic. In fact its linearization is given by the elliptic, self-adjoint Jacobi operator

$$L[\psi] = div((D^2F + F1)\nabla\psi) + \psi\langle d\nu, (D^2F + F1)\circ d\nu\rangle.$$

Later we will need the following lemma

Lemma 2. *Let $X : \Sigma \longrightarrow \mathbf{R}^{n+1}$ be an oriented, smooth, critical immersion of $J_{F,\Lambda}$. Then*

$$(n+1)\Lambda V(X) = -n\int_\Sigma F dA. \tag{9}$$

Proof. Compute

$$div(F\nabla|X|^2/2) = \langle\nabla F, \nabla|X|^2/2\rangle + F\Delta|X|^2/2. \tag{10}$$

A well-known formula gives

$$\Delta|X|^2/2 = nh\sigma + n, \tag{11}$$

where $\sigma = \langle X, \nu\rangle$ is the support function. Therefore

$$\begin{aligned} div(F\nabla|X|^2/2) &= \langle \nabla F, \nabla|X|^2/2\rangle + nh\sigma F + nF \\ &= \langle d\nu(DF), \nabla|X|^2/2\rangle + nh\sigma F + nF \\ &= \langle DF, d\nu(\nabla|X|^2/2)\rangle + nh\sigma F + nF \\ &= \langle DF, \nabla\sigma\rangle + nh\sigma F + nF, \end{aligned}$$

using the fact that $d\nu(\nabla|X|^2/2) = \nabla\sigma$. On the other hand,

$$div(\sigma DF) = \langle\nabla\sigma, DF\rangle + \sigma div DF = \langle\nabla\sigma, DF\rangle + nh\sigma F - \Lambda\sigma.$$

It then follows that

$$0 = \int_\Sigma div(F(\nabla|X|^2/2) - \sigma DF)dA = \int_\Sigma (nF + \Lambda\sigma)dA = n\int_\Sigma F dA + (n+1)V(X).$$

□

3. Main result

Let $X : \Sigma \longrightarrow \mathbf{R}^{n+1}$ be as above. We first consider the variation of X given by

$$X_t = X + t\phi(\nu) = X + t(F\nu + DF). \tag{12}$$

We then determine a function $s(t)$ such that

$$\tilde{X}_t := s(t)X_t$$

satisfies

$$\tilde{X}_0 = X, \quad V(\tilde{X}_t) \equiv const.$$

We then show that $\partial^2_{tt}J_F(\tilde{X}_t) \le 0$ holds with equality if and only if the image of X is a multiple of W_F.

Clearly,

$$J_F(\tilde{X}_t) = J_F(sX_t) = s^n J(X_t),$$

so that

$$\begin{aligned} \partial^2_{tt}J_F(\tilde{X}_t)_{t=0} =& (ns''(0) + n(n-1)(s'(0))^2)J_F(X) \\ &+ 2ns'(0)\partial_t J_F(X)_{t=0} + \partial^2_{tt}J_F(X_t)_{t=0}. \end{aligned} \tag{13}$$

Note that

$$dX_t = dX + td(\phi\circ\nu) = dX + t((D^2F + F1)\circ d\nu) \perp \nu. \tag{14}$$

This implies that if ν_t is the Gauss map of X_t then $\nu_t \equiv \nu$. It follows that

$$\partial_t J_F(X)_{t=0} = \int_\Sigma F(\nu)\partial_t(dA_{X_t})_{t=0} \tag{15}$$

and

$$\partial^2_{tt}J_F(X)_{t=0} = \int_\Sigma F(\nu)\partial_{tt}(dA_{X_t})_{t=0}. \tag{16}$$

Observe that since $\partial_t(X_t)_{t=0} = DF + F\nu$, we have

$$\partial_t(dA_{X_t})_{t=0} = (divDF - nhF)dA = -\Lambda dA. \tag{17}$$

In the following lemma we will compute the coeficients in (13).

Lemma 3. *$s'(0) = \Lambda/n$, and $s''(0) = 2\Lambda^2/n^2$.*

Proof. Recall that $s(t)$ is defined by the condition

$$V(\tilde{X}_t) = s^{n+1}(t)V(X_t) \equiv V(X). \tag{18}$$

Differentiating this gives

$$(n+1)s'(t)s^n(t)V(X_t) + s^{n+1}(t)\partial_t V(X_t) \equiv 0. \tag{19}$$

At any time t, the first variation of the volume is

$$\partial_t V(X_t) = \int_\Sigma \langle \partial_t X, \nu_t\rangle dA_{X_t} = \int_\Sigma F(\nu) dA_{X_t}, \tag{20}$$

since the normal is constant in t. In particular, when $t = 0$ this gives

$$\partial_t V(X_t)_{t=0} = \int_\Sigma F(\nu) dA_X, \tag{21}$$

so $s'(0) = \Lambda/n$ by Lemma 2. Also, differentiating (19) and setting $t = 0$ gives

$$\begin{aligned}((n+1)s''(0) + n(n+1)(s'(0))^2)V(X)\\ + 2(n+1)s'(0)\partial_t V(X_t)_{t=0} + \partial^2_{tt}V(X_t)_{t=0} = 0.\end{aligned} \tag{22}$$

It then follows from Lemma 2, (20), and (17) that $s''(0) = 2\Lambda^2/n^2$. □

Using the lemma and (13), we obtain

$$\partial^2_{tt} J_F(\tilde{X}_t)_{t=0} = \int_\Sigma (F(\Lambda^2(\frac{1}{n} - 1)dA + \partial^2_{tt}(dA_{X_t})_{t=0})). \tag{23}$$

Let $\{e_i\}_{i=1,\dots,n}$ be a local orthonormal frame for the metric induced by X defined on a neighborhood in Σ. Let $g_t = (g_{i,j}(t))$ be the matrix representation of the metric induced by X_t with respect to this frame, so that locally

$$dA_{X_t} = (detg_t)^{1/2} dA_X.$$

Since $g_0 = id$ we have

$$(detg_t)'(0) = tr(g'(0)) = -2\Lambda, \tag{24}$$

$$(detg_t)''(0) = (tr(g'(0)))^2 + tr(g''(0)) - tr(g'(0))^2, \tag{25}$$

so that

$$\begin{aligned}((detg_t)^{1/2})''(0) &= -\frac{1}{4}(trg')^2 + \frac{1}{2}(trg')^2 + \frac{1}{2}(trg'') - \frac{1}{2}tr(g')^2\\ &= \Lambda^2 + \frac{1}{2}(trg'') - tr(g')^2.\end{aligned}$$

Locally, we have

$$g'_{i,j} = \langle X'_i, X_j\rangle + \langle X_i, X'_j\rangle,$$
$$g''_{i,j} = 2\langle X'_i, X'_j\rangle.$$

Let $A_{i,j} = \langle X'_i, X_j\rangle$. Then by using the fact that dX' is tangential, we have

$$tr(g')^2 = tr(A + A^t)^2 = |A + A^t|^2 \qquad tr(g'') = 2|A|^2.$$

Therefore

$$\begin{aligned}\frac{1}{2}(trg'') - \frac{1}{2}tr(g')^2 &= \frac{1}{2}(2\sum_{i,j} A^2_{i,j} - \sum_{i,j}(A_{i,j} + A_{j,i})^2\\ &= -\sum_{i,j}(A_{i,j}A_{j,i}).\end{aligned}$$

Therefore in (23) we have, locally,

$$F(\Lambda^2(\frac{1}{n}-1)dA+\partial^2_{tt}(dA_{X_t})_{t=0}=\Lambda^2/n-\sum_{i,j}(A_{i,j}A_{j,i}). \tag{26}$$

We now choose our frame more carefully. Fix $p\in\Sigma$ and note that $dX(T_p\Sigma)=T_{\nu(p)}S^n$. We choose an orthonormal frame so that $\{e_i(p)\}$ diagonalizes $D^2F(\nu(p))$. This is possible since the Hessian is symmetric. With respect to this basis, the matrix representing $(D^2F+F1)_{\nu(p)}$ has the form $diag(\mu_1,...,\mu_n)$, with $\mu_i>0$ for all i by the convexity condition. Let $(\nu_{i,j})$ be the (symmetric) matrix representing $d\nu(p)$ with respect to the given basis. Then at p we have

$$\langle X_i',X_j\rangle=A_{i,j}=(diag(\mu_1,...,\mu_n)(d\nu))_{i,,j}=(\mu_i\nu_{i,j}).$$

Then

$$\begin{aligned}\Lambda^2/n-\sum_{i,j}(A_{i,j}A_{j,i}) &= 1/n(\sum_{i=1,\dots,n}\mu_i\nu_{i,i})^2-\sum_{i,,j=1,\dots,n}\mu_i\mu_j\nu_{i,j}^2\\ &\le (1/n)(n)\sum_{i=1,\dots,n}\mu_i^2\nu_{i,i}^2-\sum_{i=1,\dots,n}\mu_i^2\nu_{i,i}^2-\sum_{i\neq j}\mu_i\mu_j\nu_{i,j}^2\\ &= -\sum_{i\neq j}\mu_i\mu_j\nu_{ij}^2\le 0.\end{aligned}$$

Equality holds between the first two lines if and only if $\mu_i\nu_{ii}\equiv t, i=1,...,n$, for some number t. Equality holds in the last line if and only if $\nu_{i,j}=0$ for all $i\neq j$. If this is the case, then

$$\sum\mu_i\nu_{i,i}=nt=-\Lambda_\Sigma,$$

so t does not depend on the point. Therefore the integrand in (26) is non-positive and vanishes identically if and only if at each point

$$(D^2F+F1)_\nu d\nu\equiv-(\Lambda_\Sigma/n)1. \tag{27}$$

However, this implies that Σ is convex and in particular that ν is a diffeomorphism of Σ onto S^n. Let σ denote the support function of Σ and let $\tilde\sigma:=\sigma\circ\nu^{-1}$. Then the map ν^{-1} is a reparametrization of $X(\Sigma)$, and it can be expressed as

$$\nu^{-1}:=(D\tilde\sigma+\tilde\sigma\nu):S^n\longrightarrow\mathbf{R}^n. \tag{28}$$

Therefore we have

$$1=d\nu^{-1}\circ d\nu=(D^2\tilde\sigma+\tilde\sigma1)\circ d\nu.$$

It then follows from (27) that

$$D^2(\tilde\sigma+F/\Lambda)+(\tilde\sigma+F/\Lambda)1\equiv 0,$$

so that integrating gives

$$D(\tilde\sigma+F/\Lambda)+(\tilde\sigma+F/\Lambda)\nu\equiv\vec{c}=const.$$

The conclusion then follows.

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