

REAL FORMS OF A RIEMANN SURFACE OF EVEN GENUS

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(Communicated by Linda Keen)

ABSTRACT. Natanzon proved that a Riemann surface X of genus $g \geq 2$ has at most $2(\sqrt{g} + 1)$ conjugacy classes of symmetries, and this bound is attained for infinitely many genera g . The aim of this note is to prove that a Riemann surface of even genus g has at most four conjugacy classes of symmetries and this bound is attained for an arbitrary even g as well. An equivalent formulation in terms of algebraic curves is that a complex curve of an even genus g has at most four real forms which are not birationally equivalent.

1. INTRODUCTION

Natanzon [4] (see also [3]) proved that a Riemann surface X of genus $g \geq 2$ has at most $2(\sqrt{g} + 1)$ conjugacy classes of symmetries, and this bound is attained for infinitely many odd genera g . Singerman [5] showed that if X is hyperelliptic, then the number of non-conjugate pairs of symmetries does not exceed 3 if g is even and 4 if g is odd. The aim of this note is to prove that a Riemann surface of even genus g has at most 4 conjugacy classes of symmetries and this bound is attained for an arbitrary even g as well. An equivalent formulation is that a Riemann surface of even genus g is the complex double of at most 4 bordered Klein surfaces, or in terms of algebraic curves, that a complex curve of an even genus g has at most four real forms which are not birationally equivalent (see [1]).

2. PRELIMINARIES

Let X be a Riemann surface of genus $g \geq 2$. By a symmetry of X we mean an anticonformal involution σ of X with fixed points and a surface admitting a symmetry is said to be *symmetric*. A symmetric Riemann surface X corresponds to a real algebraic curve. In the group $\text{Aut}^\pm(X)$, of all conformal and anticonformal automorphisms of X , non-conjugate symmetries correspond to different real models of the curve.

Arbitrary compact Riemann surfaces of genus $g \geq 2$ can be represented as the orbit space \mathcal{H}/Γ of the hyperbolic plane \mathcal{H} with respect to the action of a Fuchsian surface group Γ , a discrete subgroup of $\text{Aut}^+(\mathcal{H}) = \text{PSL}(2, \mathbf{R})$ without elliptic elements. A discrete subgroup Λ of $\text{Aut}^\pm(\mathcal{H})$ with compact orbit space is called an

Received by the editors April 14, 1997.

1991 *Mathematics Subject Classification*. Primary 20F10, 30F10; Secondary 30F35, 51M10, 14H99.

The second author was partially supported by The Swedish Natural Science Research Council (NFR).

NEC (non-euclidean crystallographic) group. The algebraic structure of an NEC group Λ is determined by the signature:

$$(1) \quad s(\Lambda) = (h; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}).$$

The orbit space \mathcal{H}/Λ is an orbifold with underlying surface of genus h , having r cone points and k boundary components, each with $s_j \geq 0$ corner points. The signs “+” and “−” correspond to orientable and non-orientable orbifolds respectively. The integers m_i are called the proper periods of Λ and they are the orders of the cone points of \mathcal{H}/Λ . The brackets $(n_{i1}, \dots, n_{is_i})$ are the period cycles of Λ and the integers n_{ij} are the link periods of Λ ; they are the orders of the corner points of \mathcal{H}/Λ . Λ is called the *group* (or *fundamental group*) of the orbifold \mathcal{H}/Λ .

A group Λ with signature (1) has the presentation with generators: $x_1, \dots, x_r, e_1, \dots, e_k, c_{ij}, 1 \leq i \leq k, 0 \leq j \leq s_i$ and $a_1, b_1, \dots, a_h, b_h$ if \mathcal{H}/Λ is orientable or d_1, \dots, d_h otherwise, and relators: $x_i^{m_i}, i = 1, \dots, r, c_{ij-1}^2, c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, c_{i0}e_i^{-1}c_{is_i}e_i, i = 1, \dots, k, j = 0, \dots, s_i$ and $x_1 \dots x_r e_1 \dots e_k a_1 b_1 a_1^{-1} b_1^{-1} \dots a_h b_h a_h^{-1} b_h^{-1}$ or $x_1 \dots x_r e_1 \dots e_k d_1^2 \dots d_h^2$, according to whether \mathcal{H}/Λ is orientable or not.

The hyperbolic area of the orbifold \mathcal{H}/Λ coincides with the hyperbolic area of an arbitrary fundamental region of Λ and equals:

$$(2) \quad \mu(\Lambda) = 2\pi(\varepsilon h - 2 + k + \sum_{i=1}^r (1 - \frac{1}{m_i}) + \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^{s_i} (1 - \frac{1}{n_{ij}})),$$

where $\varepsilon = 2$ if there is a “+” sign and $\varepsilon = 1$ otherwise. If Λ' is a subgroup of Λ of finite index, then it is an NEC group itself and the following Riemann-Hurwitz formula holds:

$$(3) \quad [\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda).$$

Given an NEC group Λ the subgroup Λ^+ of Λ consisting of the orientation-preserving elements is called the *canonical Fuchsian subgroup* of Λ .

An NEC group Γ without elliptic elements is called a *surface group* and it has signature $(h; \pm; [-], \{(-), \dots, (-)\})$. In such a case \mathcal{H}/Γ is a *Klein surface*, i.e., a surface with a dianalytic structure of topological genus h , orientable or not according to whether the sign is “+” or “−” and having k boundary components. Conversely, a Klein surface whose complex double has genus greater than one can be expressed as \mathcal{H}/Γ for some NEC surface group Γ . Furthermore, given a Riemann (resp. Klein) surface represented as the orbit space $X = \mathcal{H}/\Gamma$, with Γ a surface group, a finite group G is a group of automorphisms of X if and only if $G = \Lambda/\Gamma$ for some NEC group Λ .

3. 2-GROUPS OF AUTOMORPHISMS OF SURFACES OF EVEN GENUS

We start this section with the following lemma which allows us to restrict ourselves to finite 2-groups when we work with conjugacy classes of symmetries of a Riemann surface.

Lemma 3.1. *Let g be a positive integer greater than or equal to 2. There exists a Riemann surface X of genus g having k non-conjugate symmetries which generate a 2-subgroup of $\text{Aut}^\pm(X)$ such that any Riemann surface X' of genus g has no more than k conjugacy classes of symmetries.*

Proof. Let X be a Riemann surface of genus g with maximal number k of non-conjugate symmetries τ_1, \dots, τ_k and let G_2 be a 2-Sylow subgroup of $G = \text{Aut}^\pm(X)$. Then by the Sylow theorem $\tau_1^{\alpha_1}, \dots, \tau_m^{\alpha_m} \in G_2$, for some $\alpha_1, \dots, \alpha_m \in G$, and hence the result.

Lemma 3.2. *Let X be a Riemann surface of even genus g and let G be a 2-group of automorphisms of X . Then G is an extension of a cyclic or dihedral group by Z_2 . In particular, if G is generated by symmetries, then either G is a cyclic group, a dihedral group, or a semidirect product of a cyclic or dihedral group by Z_2 .*

Proof. Let $X = \mathcal{H}/\Gamma$ for some surface NEC group Γ and let $G = \Lambda/\Gamma$. Assume that Λ has a signature of a general form (1). By Theorems 2.2.4 and 2.3.3 of [2], G is cyclic or dihedral if Λ has a proper period is equal to $|G|$ or a link period equal to $|G|/2$ respectively. So assume that neither a proper period of Λ is equal to $|G|$ nor a link period is equal to $|G|/2$. Let $M = \{i \mid 2m_i = |G|\}$ and $N = \{(i, j) \mid 4n_{ij} = |G|\}$, and let N' and M' be the complementary sets. Then

$$\frac{|G|}{2} \left(\alpha h - 2 + k + \sum_{i \in M'} \left(1 - \frac{1}{m_i}\right) + \sum_{(i,j) \in N'} \frac{1}{2} \left(1 - \frac{1}{n_{ij}}\right) \right)$$

is even and, since g is even, we obtain from the Riemann-Hurwitz formula (3) that

$$\frac{|G|}{2} \left(\sum_{i \in M} \left(1 - \frac{1}{m_i}\right) + \sum_{(i,j) \in N} \frac{1}{2} \left(1 - \frac{1}{n_{ij}}\right) \right)$$

is odd. But the last is equal to $|G||M|/2 + |G||N|/4 - |M| - |N|$ and therefore $|M| + |N|$ is odd. So in particular $|M| \neq 0$ or $|N| \neq 0$. In the first case G contains $H = Z_{|G|/2}$ while in the second one $H = D_{|G|/4}$ as a subgroup of index 2, hence the first part of the theorem. Now, if in addition G is generated by elements of order 2, then there exists an element $g \in G \setminus H$ of order 2 and so $G = H \rtimes Z_2$. This completes the proof.

Theorem 3.3 (Main result). *A Riemann surface of even genus g has at most 4 conjugacy classes of symmetries. Furthermore this bound is attained for every even genus g .*

Proof. The dihedral 2-group has three conjugacy classes of elements of order 2 and it is easy to check that all semidirect products $G = Z_n \rtimes Z_2$ have at most three such classes. So it remains to count the number of conjugacy classes of symmetries of Riemann surfaces whose groups of automorphisms are semidirect products $G = D_n \rtimes Z_2$, where n is a power of 2. These groups have the presentations:

$$G_{\alpha,\beta} = \langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^n = 1, zxz = (xy)^\alpha x, zyz = (xy)^\beta x \rangle,$$

where $\alpha - \beta \equiv 1 \pmod{2}$, $\alpha(\alpha - \beta + 1) \equiv 0 \pmod{n}$, $\beta(\alpha - \beta) + \alpha + 1 \equiv 0 \pmod{n}$.

From the proof of the previous lemma it follows that we can assume that x, y and z are symmetries of X . So neither $(xy)^{n/2}$ nor an element of the form $z(xy)^\delta x$ can be a symmetry since they preserve the orientation of X as compositions of an even number of symmetries.

Finally an element of the form $z(xy)^\gamma$ has order two if and only if

$$\gamma(\alpha - \beta + 1) \equiv 0 \pmod{n}.$$

We have

$$(xy)^m z(xy)^{-m} = z(xy)^{m(\alpha-\beta-1)} \text{ and } x(xy)^m z(xy)^{-m} x = z(xy)^{m(\beta+1-\alpha)+\alpha}$$

and so $z(xy)^{2s\alpha}$ and $z(xy)^{(2s+1)\alpha}$ are conjugate to z for all s . In particular for α odd there is only one conjugacy class of elements of order 2 of the form $z(xy)^\gamma$.

Now let α be even and let $\alpha - \beta + 1 = 2^s t$, where t is odd.

If $s = 0$, then $\alpha - \beta + 1 = 0$ and so arbitrary $z(xy)^\gamma$ has order 2. Furthermore, $(xy)^{-m} z(xy)^m = z(xy)^{2m}$, $(xy)^{-m} z(xy)(xy)^m = z(xy)^{2m+1}$ and so there are at most two conjugacy classes of elements of order 2 of this form with representatives z and zxy . But actually these elements are non-conjugate in G since also $x(xy)^m z(xy)^m x^{-1} = z(xy)^{2m+\alpha}$, where $2m + \alpha$ is even. If $s = 1$, then $\alpha = n/2$ or $\alpha = 0$ and $z, z(xy)^{n/2}$ are the only elements of order 2 of this form and furthermore they are non-conjugate just for $\alpha = 0$ and $\beta = n - 1$, i.e., for $G = D_n \times Z_2$. Indeed for $\alpha \neq 0$, $xzx = z(xy)^\alpha$ and, for $\alpha = 0, \beta \neq n - 1$, $yz y = z(xy)^{\beta+1}$. Finally, if $s \geq 2$, then every element of order 2 of this form is equal to $z(xy)^{2v}$ for some v . On the other hand $\alpha - \beta - 1 = 2u$, for some odd u , and thus every element of the form $z(xy)^{2v}$ is conjugate to z . This completes the proof of the first part.

To prove the second part of the theorem, observe that by [6] the signature of the canonical Fuchsian subgroup of an NEC-group with signature $(0; +; [-]; \{(2, q+3, 2)\})$ is maximal. So by Remark 5.1.1(1) and Theorem 5.1.2 of [2] there exists a maximal NEC-group Λ with this signature. Let $\theta : \Lambda \rightarrow G = Z_2 \times Z_2 \times Z_2 = \langle x, y, z \rangle$ defined by $\theta(c_{2j}) = x$, $\theta(c_{2j+1}) = y$ for $0 \leq j \leq (g-2)/2$, $\theta(c_g) = x$, $\theta(c_{g+1}) = z$ and finally $\theta(c_{g+2}) = zxy$. Then by [2], $\Gamma = \ker \theta$ is a surface Fuchsian group, and by the Riemann-Hurwitz formula $X = \mathcal{H}/\Gamma$ is a surface of genus g . Finally its symmetries x, y, z and zxy are non-conjugate since $G = \text{Aut}^\pm(X)$. \square

Notice that the groups $G_{0, n-1} = D_n \times Z_2$ and $G_{\alpha, \alpha+1}$, where α is even, are isomorphic. In fact, the application

$$\varphi(x) = x, \quad \varphi(y) = y, \quad \varphi(z) = z(xy)^{\alpha/2} x$$

induces an isomorphism $\varphi : G_{\alpha, \alpha+1} \rightarrow D_n \times Z_2$.

In this way, apart from the quantitative result concerning the number of non-conjugate symmetries obtained above, we have obtained also the following qualitative result.

Corollary 3.4. *Let X be a Riemann surface of even genus and let G be a subgroup of $\text{Aut}^\pm(X)$ generated by non-conjugate symmetries $\sigma_1, \sigma_2, \sigma_3$ and σ_4 . Then $G = D_n \times Z_2$.*

ACKNOWLEDGMENTS

The authors wish to thank Peter Turbek and the referee for their helpful comments.

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