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ON AN ANALOGUE OF SELBERG'S EIGENVALUE CONJECTURE FOR $\mathrm{SL}_3(\mathbf{Z})$

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Abstract. Let $\mathcal{H}$ be the homogeneous space associated to the group $\mathrm{PGL}_3(\mathbf{R})$. Let $X = \Gamma\backslash\mathcal{H}$ where $\Gamma = \mathrm{SL}_3(\mathbf{Z})$ and consider the first nontrivial eigenvalue λ_1 of the Laplacian on $L^2(X)$. Using geometric considerations, we prove the inequality $\lambda_1 > 3\pi^2/10$. Since the continuous spectrum is represented by the band $[1,\infty)$, our bound on λ_1 can be viewed as an analogue of Selberg's eigenvalue conjecture for quotients of the hyperbolic half space.

§1. Statement of the main theorem

A fundamental question in the spectral theory of automorphic forms is whether small eigenvalues exist. More specifically, let G be a noncompact reductive group with finite center, Γ a nonuniform lattice, K a maximal compact subgroup of G, and set $X = \Gamma\backslash G/K$. It is well known from the theory of Eisenstein series that $L^2(X)$ has continuous spectrum for the ring of invariant differential operators, and in particular for the positive Laplacian, $\mathbf{\Delta}$. The continuous spectrum will be, in cases of interest such as $\mathrm{PGL}_n(\mathbf{R})$, an interval $[a,\infty)$ with $a > 0$. The question we referred to above is: Do nonconstant square integrable eigenforms exist with eigenvalue $\lambda < a$? This problem is important for various considerations in number theory. In the case $G = \mathrm{PGL}_2(\mathbf{R})$ and Γ is a congruence subgroup, Selberg conjectured that no such nontrivial small eigenvalues exist.

In this paper, we consider the case when

$$G = \mathrm{PGL}_3(\mathbf{R}) \quad \text{and} \quad \Gamma = \mathrm{SL}_3(\mathbf{Z}).$$

Our main result is the following.

Theorem. *Let λ_1 be the eigenvalue for the first nontrivial eigenform on $L^2(X)$. Then*

$$\lambda_1 > 3\pi^2/10 > 2.96088.$$

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§2. Notation

Let $\mathcal{H} = G/K$ and set $X = \Gamma\backslash\mathcal{H}$. Explicit coordinates for $\tau \in \mathcal{H}$ via the Iwasawa decomposition are given by

$$\tau = \begin{bmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $y_1, y_2 > 0$, from which one can compute that the (positive) Laplacian $\mathbf{\Delta}$ can be written as

$$\begin{aligned} -\mathbf{\Delta} = {} & y_1^2 \frac{\partial^2}{\partial y_1^2} - y_1 y_2 \frac{\partial^2}{\partial y_1 \partial y_2} + y_2^2 \frac{\partial^2}{\partial y_2^2} + y_1^2(x_2^2 + y_2^2)\frac{\partial^2}{\partial x_3^2} \\ & + y_1^2 \frac{\partial^2}{\partial x_1^2} + y_2^2 \frac{\partial^2}{\partial x_2^2} + 2y_1^2 x_2 \frac{\partial^2}{\partial x_1 \partial x_3} \end{aligned}$$

(see pages 17 and 33 of [Bu84]). The ring of invariant differential operators is spanned by the identity, the Laplacian $\mathbf{\Delta}$, and a third order operator $\mathbf{\Delta}_3$ (see [Bu84]). The invariant volume element is given by

$$dV = \frac{dx_1 dx_2 dx_3 dy_1 dy_2}{y_1^3 y_2^3}.$$

We shall not use the explicit formula for the invariant volume element; however, the above expression for the Laplacian will be necessary in our proof of the main theorem.

For our purposes, it is more convenient to work with functions on $\mathcal{H}$ that are $\mathrm{SL}_3(\mathbf{Z})$ invariant rather than considering functions on the quotient space X. To this end, we introduce a fundamental domain for $\Gamma\backslash\mathcal{H}$. Specifically, computations on page 56 of [Gr93] show that a fundamental domain $\mathcal{D}$ is described through the following set of inequalities:

$$\begin{aligned} & v^{\frac{3}{2}} < v^{\frac{3}{2}}(1 - x_2 + x_3)^2 + w(1 - x_1)^2 + w^{-1}; \\ & v^{\frac{3}{2}} < v^{\frac{3}{2}}(x_2 - x_3)^2 + w(1 - x_1)^2 + w_1^{-1}; \\ & v^{\frac{3}{2}} < v^{\frac{3}{2}} x_2^2 + w; \quad v^{\frac{3}{2}} < v^{\frac{3}{2}} x_3^2 + w x_1^2 + w^{-1}; \\ & 1 < w^{-2} + x_1^2; \quad 0 < x_1 < \frac{1}{2}; \quad 0 < x_2 < \frac{1}{2}; \quad -\frac{1}{2} < x_3 < \frac{1}{2}, \end{aligned}$$

where we have used the notation $w^{-1} = y_1$ and $v^{-\frac{3}{2}} = y_2^2 y_1$. Let S denote the Siegel set described via the inequalities

$$0 < x_1 < \frac{1}{2}, \quad 0 < x_2 < \frac{1}{2}, \quad -\frac{1}{2} < x_3 < \frac{1}{2}, \quad y_1 > \frac{\sqrt{3}}{2}, \quad y_2 > \frac{\sqrt{3}}{2}.$$

The set S contains the fundamental domain $\mathcal{D}$. Further, results from page 61 of [Gr93] show the existence of elements $\gamma_1, ..., \gamma_{10} \in \mathrm{SL}_3(\mathbf{Z})$ such that $S \subset \bigcup_{i=1}^{10} \mathcal{D}\gamma_i$ (we have used the notation $\mathcal{D}\gamma$ to denote the image of the fundamental domain $\mathcal{D}$ under left multiplication by γ). The main aspects of the above points which we shall use are the assertions that for any $\tau \in S$ we have $y_1(\tau) > \sqrt{3}/2$ and that S is contained in ten translates of $\mathcal{D}$.

Recall that an automorphic form is a C^∞ function ϕ on $\mathcal{H}$ which satisfies the following properties:

(i) $\phi(\gamma \circ \tau) = \phi(\tau)$ for $\gamma \in \mathrm{SL}_3(\mathbf{Z})$;
(ii) $|\phi(\tau)| \ll y_1^{N_1} y_2^{N_2}$ for $\tau \in \mathcal{D}$ and integers N_1, N_2;
(iii) ϕ is an eigenform for the ring of invariant differential operators.

An automorphic form is said to be a cusp form if it satisfies the additional property

(iv) $$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi\left(\begin{bmatrix}1 & 0 & \xi_3\\ 0 & 1 & \xi_1\\ 0 & 0 & 1\end{bmatrix}\tau\right) d\xi_1 d\xi_3 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi\left(\begin{bmatrix}1 & \xi_2 & \xi_3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\tau\right) d\xi_2 d\xi_3 = 0.$$

Cusp forms are square integrable. Although we shall not need this fact, let us note that, from the theory of Eisenstein series, the only noncuspidal square integrable automorphic forms on X are constant.

§3. Proof of the main theorem

Our method of proof is a modification of that used by Roelcke to show that the small eigenvalue λ_1 for the quotient space $\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{R})/\mathrm{SO}_2(\mathbf{R})$ satisfies the bound $\lambda_1 > 3\pi^2/2$ (see page 511 of [He83]). We shall use the Fourier expansion of automorphic forms associated to $\mathrm{SL}_3(\mathbf{Z})$, as developed in Chapter IV of [Bu84].

Assume that ϕ is a nonconstant automorphic form, so then $\mathbf{\Delta}\phi = \lambda\phi$ and $\phi\mathbf{\Delta}\phi = \lambda\phi^2$. Through integration by parts, using the automorphic boundary conditions, and the fact that the Siegel domain S is contained in ten translates of the fundamental domain $\mathcal{D}$, we obtain the inequality

$$\frac{\int_S |\nabla\phi|^2 dV}{\int_S |\phi|^2 dV} < 10\lambda.$$

As on page 67 of [Bu84], let us expand ϕ in a Fourier expansion with respect to the abelian group

$$\left\{\begin{bmatrix}1 & 0 & \xi_3\\ 0 & 1 & \xi_1\\ 0 & 0 & 1\end{bmatrix} : \quad \xi_1, \xi_3 \in \mathbf{R}\right\}.$$

Specifically, we have $\phi(\tau) = \sum_{n_1, n_3} \phi_{n_1}^{n_3}(\tau)$ where

$$\phi_{n_1}^{n_3}(\tau) = \int_0^1 \int_0^1 \phi\left[\begin{pmatrix}1 & 0 & \xi_3\\ 0 & 1 & \xi_1\\ 0 & 0 & 1\end{pmatrix}\tau\right] e^{-2\pi i(n_1\xi_1 + n_3\xi_3)} d\xi_1 d\xi_3.$$

Observe that $\phi_0^0 = 0$ since ϕ is not constant and square integrable, hence cuspidal. Let

$$\Gamma_1^2 = \left\{\begin{bmatrix}a & b & 0\\ c & d & 0\\ 0 & 0 & 1\end{bmatrix} : \begin{bmatrix}a & b\\ c & d\end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z})\right\},$$

and set Γ_∞^2 to be the subgroup of Γ_1^2 which stablizes infinity. As on page 69 of [Bu84], we then can write $\phi_{n_1}^{n_3}$ as

$$\phi_{n_1}^{n_3}(\tau) = \sum_{\gamma \in \Gamma_\infty^2 \backslash \Gamma_1^2} \sum_{n_1=1}^{\infty} \phi_{n_1}^0(\gamma \circ \tau).$$

By a standard application of elliptic regularity, ϕ is necessarily C^∞, hence we can interchange integration and summation and apply Parseval's theorem to obtain the inequality

$$\frac{\int_S \sum_{n_1=1}^{\infty} \left| \sum_{\gamma \in \Gamma_\infty^2 \backslash \Gamma_1^2} \nabla_\tau \phi_{n_1}^0(\gamma \circ \tau) \right|^2 dV}{\int_S \sum_{n_1=1}^{\infty} \left| \sum_{\gamma \in \Gamma_\infty^2 \backslash \Gamma_1^2} \phi_{n_1}^0(\gamma \circ \tau) \right|^2 dV} < 10\lambda.$$

Since ∇ is an invariant operator, we may differentiate the expressions in the numerator and then evaluate at $\gamma \circ \tau$, thus yielding

$$\frac{\int_S \sum_{n_1=1}^{\infty} \left| \sum_{\gamma \in \Gamma_\infty^2 \backslash \Gamma_1^2} \nabla_\tau \phi_{n_1}^0(\tau)\big|_{\gamma\circ\tau} \right|^2 dV}{\int_S \sum_{n=1}^{\infty} \left| \sum_{\gamma \in \Gamma_\infty^2 \backslash \Gamma_1^2} \phi_{n_1}^0(\tau)\big|_{\gamma\circ\tau} \right|^2 dV} < 10\lambda.$$

We now integrate by parts and consider the action of the Laplacian $\mathbf{\Delta}$ on functions of the form $\phi_{n_1}^0$. Since each function $\phi_{n_1}^0$ is independent of x_3, these terms in $\mathbf{\Delta}$ annihilate $\phi_{n_1}^0$. Observe that all terms involving y_1, y_2 and x_2 are positive operators (compare with line (2.31) on page 32 of [Bu84]), so we obtain the bound

$$\mathbf{\Delta}\phi_{n_1}^0 \geq -y_1^2 \cdot \frac{\partial^2}{\partial x_1^2}\phi_{n_1}^0 = y_1^2 \cdot 4\pi^2 n_1^2 \phi_{n_1}^0.$$

Since $y_1^2 > \frac{3}{4}$, we have $\mathbf{\Delta}\phi_{n_1}^0 \geq \frac{3}{4}\cdot 4\pi^2 n_1^2 \phi_{n_1}^0 = 3\pi^2 n_1^2 \phi_{n_1}^0$. Combining this inequality with the above calculations and the cuspidality condition $\phi_0^0 = 0$, we obtain

$$10\lambda > \frac{\int_S \sum_{n_1=1}^{\infty} \sum_{\gamma \in \Gamma_\infty^2 \backslash \Gamma_1^2} \mathbf{\Delta}_\tau \phi_{n_1}^0(\tau)\big|_{\gamma\circ\tau} \cdot \phi_{n_1}^0(\tau)\big|_{\gamma\circ\tau} dV}{\int_S \sum_{n=1}^{\infty} \left| \sum_{\gamma \in \Gamma_\infty^2 \backslash \Gamma_1^2} \phi_{n_1}^0(\tau)\big|_{\gamma\circ\tau} \right|^2 dV} \geq 3\pi^2,$$

hence $\lambda \geq 3\pi^2/10$. Since ϕ was any cusp form, we obtain the bound as asserted in the theorem.

§4. Concluding remarks

As the continuous spectrum in this situation is $[1, \infty)$, our theorem implies an analogue of Selberg's eigenvalue conjecture. Note that our bound is stronger than the result for $\mathrm{SL}_3(\mathbf{Z})$ from [Mi96] where it was proved that $\lambda_1 \geq 1$. In general, our method applies to $G = \mathrm{SL}_n(\mathbf{R})$ with $\Gamma = \mathrm{SL}_n(\mathbf{Z})$ to give the bound $\lambda_1 > 3\pi^2/M$ where M is the number of fundamental domains which intersect a Siegel set containing the fundamental domain constructed in [Gr93]; however, for $n \geq 4$, this bound is rather weak. Finally, let us remark that our theorem is indeed a consequence of the Ramanujan conjecture, which asserts that all nontrivial automorphic representations come from tempered representations.

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