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POSITIVE SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS IN THE EUCLIDEAN PLANE

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Abstract. In the present paper, we study the existence of solutions to the problem

$$\begin{cases} \Delta u + f(x,u) = 0 & \text{in } D \\ u > 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

where D is an unbounded domain in $\mathbb{R}^2$ with a compact nonempty boundary ∂D consisting of finitely many Jordan curves. The goal is to prove an existence theorem for the above problem in a general setting by using Brownian path integration and potential theory.

1. Introduction

In the present paper, we study the existence of solutions to the problem

$$\begin{cases} \Delta u + f(x,u) = 0 & \text{in } D \\ u > 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \tag{1.1}$$

where D is an unbounded domain in $\mathbb{R}^2$ with a compact nonempty boundary ∂D consisting of finitely many Jordan curves. The goal is to prove an existence theorem for problem (1.1) in a general setting by using Brownian path integration and potential theory.

For $D \subset \mathbb{R}^n$, $n \geq 3$, D unbounded and with a compact Lipschitz boundary, Z. Zhao [13] proved the following problem:

$$\begin{cases} \Delta u + K(x)f(u) = 0 & \text{in } D \\ u > 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \tag{1.2}$$

if K is a Borel measurable function in D satisfying that the family $\{\frac{K(.)}{|.-x|^{d-2}}\}$ is uniformly integrable over D with a parameter $x \in D$, and f is a continuous function in $(0, b)$ for some $0 < b \leq \infty$ satisfying that:

$$\lim_{w \to 0^+} \frac{f(w)}{w} = 0.$$

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Then the problem (1.2) has infinitely many bounded solutions. More precisely, there exists $b_0 \in (0, b]$ such that for each $c \in (0, b_0]$ there exists a solution u of (1.2) satisfying

$$\lim_{|x|\to\infty} u(x) = c.$$

A similar nonlinear problem for the second order ordinary differential equations is studied in [14]. We state our main result as

Theorem 1.1 (The Main Theorem). *Let D be an unbounded domain in $\mathbb{R}^2$ with a compact nonempty boundary ∂D consisting of finitely many Jordan curves. Suppose that $f(x,s)$ is a Borel measurable function in $\mathbb{R}^2 \times \mathbb{R}^+$ and $f(x,\cdot)$ is a continuous function in $\mathbb{R}^+$ for each fixed $x \in \mathbb{R}^2$. If there exists a positive, convex and continuously differentiable function $F(x,s)$ in $\mathbb{R}^2 \times \mathbb{R}^+$ satisfying the conditions*

$$|f(x,s)| \le F(x,s), \quad (x,s) \in \mathbb{R}^2 \times \mathbb{R}^+, \tag{1.3}$$

$$F(x,0) = F_s(x,0) = 0, \tag{1.4}$$

and

$$F_s(x, \ln(|x|+1)) \in K_2^\infty, \tag{1.5}$$

then the problem

$$\begin{cases} \Delta u + f(x,u) = 0 & \text{in } D \\ u > 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \tag{1.6}$$

has infinitely many solutions. More precisely there exists a number $b > 0$ such that for each $c \in (0, b]$, (1.6) has a solution u satisfying

$$\lim_{|x|\to\infty} \frac{u(x)}{\ln|x|} = c. \tag{1.7}$$

Remark. A special case of the above general setting is the semilinear problem for any V in K_2, and $p > 1$: If we take $f(x,s) = V(x)s^p$ and $F(x,s) = |V(x)|s^p$, then problem (1.1) becomes the corresponding problem for the semilinear elliptic equation

$$\begin{cases} \Delta u + V(x)u^p = 0 & \text{in } D \\ u > 0 & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases} \tag{1.8}$$

Then f and F satisfy (1.3) and (1.4) obviously. Condition (1.5) on F is equivalent to

$$\int_{|x|\ge 1} |V(x)|[\ln(1+|x|)]^p dx < \infty.$$

2. Green functions and h-conditional Brownian motion

In this section we collect the results on Green functions in $\mathbb{R}^2$ and h-conditional Brownian motion.

The Kato class K_2 is defined to be the set of all Borel measurable functions on $\mathbb{R}^2$ satisfying

$$\lim_{\alpha\downarrow 0}\left[\sup_{x\in\mathbb{R}^2}\int\limits_{|y-x|<\alpha}\ln\frac{1}{|y-x|}|q(y)|dy\right]=0 \tag{2.1}$$

where $q:\mathbb{R}^2\to\mathbb{R}$ is measurable. K_2 is an important class of the potential functions for the Schrödinger operator. Another potential class K_2^∞ which was introduced in [5] is as follows:

$$K_2^\infty=\left\{q\in K_2:\int\limits_{|y|>1}\ln|y||q(y)|dy<\infty\right\}. \tag{2.2}$$

Let $\{X_t\}$ be 2-dimensional Brownian motion (see [2]). For any open or closed set A in $\mathbb{R}^2$, let

$$T_A(\omega)=\inf\{t>0:X_t(\omega)\in A\}.$$

This is called the hitting time of A. The hitting time of A^c is called the exit time from A and denoted by τ_A. We shall use P^x to denote the probability measure on the Brownian continuous paths starting at x. E^x is the expectation on P^x. Let $G_D(x,y)$ be the Green function for D. We list some known results.

For any Borel function φ in D the Green operator is defined as

$$\begin{aligned}G_D\varphi(x)&=E^x\left[\int_0^{\tau_D}\varphi(X_t)dt\right]\\&=\int\limits_D G_D(x,y)\varphi(y)dy,\end{aligned}$$

and

$$\Delta(G_D\varphi)=-2\varphi. \tag{2.3}$$

Proposition 2.1. *Let D be an unbounded domain in $\mathbb{R}^2$ with a compact nonempty boundary ∂D consisting of finitely many Jordan curves. Then there exists a harmonic function $h>0$ in D such that*

$$\lim_{D\ni x\to z}h(x)=0\quad\forall z\in\partial D, \tag{2.4}$$

and

$$\lim_{|x|\to\infty}\frac{h(x)}{\ln|x|}=1. \tag{2.5}$$

Proof. Pick a point $a\in\mathbb{R}^2\backslash\bar{D}$ and $r>0$ (small enough) such that $\bar{B}(a,r)\subset\mathbb{R}^2\backslash\bar{D}$. Let $x^*=a+r^2\frac{(x-a)}{|x-a|^2}$ be the Kelvin inversion from $D\cup\{\infty\}$ onto D^*, where $D^*=\{x^*\in B(a,r):x\in D\cup\{\infty\}\}$. Since the Kelvin inversion preserves harmonic functions and D^* is bounded Jordan domain, we have

$$G_D(x,y)=G_{D^*}(x^*,y^*),\quad\forall x,y\text{ in }D.$$

Let

$$h(x)=\pi G_{D^*}(x^*,a).$$

Thus, we have

$$\lim_{y\to\infty} \pi G_D(x,y) = \lim_{y^*\to a} \pi G_{D^*}(x^*,y^*) = \pi G_{D^*}(x^*,a). \tag{2.6}$$

The rest of the proof is just the properties of the Green function in the bounded domain. □

Now we can develop the notion of conditional Brownian motion introduced by Doob for a general theory. Let $h > 0$ be a harmonic function in D. We define

$$p_h^D(t;x,y) = h(x)^{-1} p^D(t;x,y) h(y), \quad t > 0, \quad x,y \in D,$$

where p^D is the transition density of the killed Brownian motion. $p_h^D(t;x,y)$ satisfies the conditions for a transition density. Let P_h^x and E_h^x denote respectively the probability and expectation determined by the h-conditional Brownian motion starting at x. Then we have for all $f \in \mathcal{B}^+(D)$:

$$E_h^x[t < \tau_D; f(X_t)] = h(x)^{-1} E^x[t < \tau_D; f(X_t) h(X_t)], \quad x \in D.$$

Theorem 2.2 (3-G)**.** *Let $D \subset \mathbb{R}^2$ be as in Proposition 2.1. Then there exists a constant $\mathbf{C} > 0$ depending on D only such that $\forall x, y,$ and $z \in D$:*

$$\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \le \mathbf{C}[G_D(x,y) + G_D(y,z) + 1]. \tag{2.7}$$

Proof. We used Kelvin transformation to reduce to the bounded Jordan domain case (see [2]). For $n \ge 3$, the 3-G theorem for the unbounded Lipschitz domain was proved by Herbst and Zhao [6] based on [4]. □

Proposition 2.3. *If $q \in K_2^\infty$, then $q \in L^1(\mathbb{R}^2)$.*

Proof. Let $q \in K_2^\infty$. Then by definition of K_2^∞, $q \in K_2$. So there exists $\alpha_1 > 0$ such that

$$\sup_{x\in\mathbb{R}^2} \int\limits_{|y-x|\le\alpha_1} \ln\frac{1}{|x-y|}|q(y)|dy < 1 \tag{2.8}$$

and

$$\int\limits_{|y|\ge e} \ln|y||q(y)|dy < \infty. \tag{2.9}$$

Let $\alpha = \min\left(\alpha_1, e^{-1}\right)$. For the compact set $\bar{B} = \bar{B}(0,e)$, there exists finite points x_i, $i = 1, \dots, L$, such that $\bar{B} \subseteq \bigcup_{i=1}^{L} B(x_i,\alpha)$ and $\forall i = 1, \dots, L$, $y \in B(x_i,\alpha)$ we have $\ln\frac{1}{|x_i-y|} \ge \ln\frac{1}{e^{-1}} = 1$. Then by (2.8)

$$\begin{aligned} \int\limits_{|y|\le e} |q(y)|dy &\le \sum_{i=1}^{L} \int\limits_{B(x_i,\alpha)} |q(y)|dy \\ &\le \sum_{i=1}^{L} \int\limits_{B(x_i,\alpha)} \ln\frac{1}{|x_i-y|}|q(y)|dy \\ &\le L. \end{aligned} \tag{2.10}$$

Since $q \in K_2^\infty$, we have

$$\int\limits_{|y|>e} |q(y)|dy \leq \int\limits_{|y|>e} \ln|y||q(y)|dy < \infty. \tag{2.11}$$

Thus, by (2.10) and (2.11)

$$\int\limits_{\mathbb{R}^2} |q(y)|dy \leq \int\limits_{|y|>e} |q(y)|dy + L < \infty.$$

□

3. Uniformly integrable functions

This is the main technical section in which we investigate uniform integrability of the family of the functions $\{G_D(x,.)|q(.)|\}$ and $\{\frac{1}{h(x)}G_D(x,.)h(.)|q(.)|\}$.

Since $D \supset B_r^* = \mathbb{R}^2\backslash\overline{B(a,r)}$, using the explicit formula of $G_{B_r^*}(.,.)$ and the definition of K_2^∞, we can prove:

Proposition 3.1. *For $q \in K_2^\infty$, the family of functions $\{G_D(x,\cdot)|q(\cdot)|\}$ with parameter $x \in D$ is uniformly integrable over D.*

We define

$$\|q\|_D = \sup_{x\in D} \int\limits_D G(x,y)|q(y)|dy, \tag{3.1}$$

and

$$\|q\|_1 = \int\limits_D |q(y)|dy. \tag{3.2}$$

Proposition 3.2. *Let D and h be as in Proposition 2.1. Then for any $q \in K_2^\infty$, we have*

(a) *$\{\frac{1}{h(x)}G_D(x,\cdot)h(\cdot)|q(\cdot)| : x \in D\}$ is uniformly integrable.*
(b) *There exists a constant c_D such that $\forall x \in D$:*

$$\frac{1}{h(x)}\int\limits_D G_D(x,y)h(y)|q(y)|dy \leq c_D(\|q\|_D + \|q\|_1). \tag{3.3}$$

Proof. By Theorem (3-G), there exists $\mathbf{C} > 0$ such that

$$\frac{G_D(x,y)G_D(y,z)|q(y)|}{G_D(x,z)} \leq \mathbf{C}\left(G_D(x,y) + G_D(y,z) + 1\right)|q(y)|. \tag{3.4}$$

Recalling $h(x) = \lim_{z\to\infty} \pi G_D(x,z)$, and using Proposition 3.1 and the fact $q \in L^1(D)$, we can obtain that $\{\frac{1}{h(x)}G_D(x,\cdot)h(\cdot)|q(\cdot)| : x \in D\}$ is uniformly integrable. Thus, (a) follows.

By taking the integral of (3.4) with respect to y, we have

$$\begin{aligned}
&\frac{1}{G_D(x,z)}\int_D G_D(x,y)G_D(y,z)|q(y)|dy \\
&\le \mathbf{C}\Big(\int_D G_D(x,y)|q(y)|dy + \int_D G_D(y,z)|q(y)|dy + \int_D |q(y)|dy\Big) \\
&\le c_D(\|q\|_D + \|q\|_1),
\end{aligned} \tag{3.5}$$

where $c_D = 2\mathbf{C}$. By taking the limit of (3.5) as $z \to \infty$, we have by Fatou's lemma:

$$\frac{1}{h(x)}\int_D G_D(x,y)h(y)|q(y)|dy \le c_D(\|q\|_D + \|q\|_1).$$

□

Proposition 3.3. *Let D and h be as in Proposition 2.1. Let $q \in K_2^\infty$. If*

$$\sup_{x\in D} E_h^x\left[\int_0^{\tau_D} |q(X_t)|dt\right] \le \frac{1}{2}, \tag{3.6}$$

then the function

$$w(x) = h(x)E_h^x\left[e^{\int_0^{\tau_D} q(X_s)ds}\right], \quad \forall x \in D, \tag{3.7}$$

is well defined in D and satisfies that $\forall x \in D$

$$e^{-\frac{1}{2}}h(x) \le w(x) \le 2h(x), \tag{3.8}$$

and

$$w(x) = h(x) + \int_D G_D(x,y)q(y)w(y)dy. \tag{3.9}$$

Proof. By Khas'minskii's lemma, we have by (3.6),

$$E_h^x\left[e^{\int_0^{\tau_D} q(X_s)ds}\right] \le 2. \tag{3.10}$$

By Jensen's inequality and (3.6), we have

$$e^{-\frac{1}{2}} \le E_h^x\left[e^{\int_0^{\tau_D} q(X_s)ds}\right]. \tag{3.11}$$

Thus, it follows by (3.7), (3.10), and (3.11) that

$$h(x)e^{-\frac{1}{2}} \le w(x) \le 2h(x).$$

The equality (3.9) will be obtained by calculation on the conditional Brownian paths:

$$\begin{aligned}
w(x) - h(x) &= h(x) E_h^x \left[e^{\int_0^{\tau_D} q(X_s)ds} - 1 \right] \\
&= h(x) E_h^x \left[\int_0^{\tau_D} q(X_t) e^{\int_t^{\tau_D} q(X_s)ds} dt \right] \\
&= h(x) \int_0^\infty E_h^x \left[t < \tau_D : q(X_t) e^{\int_t^{\tau_D} q(X_s)ds} \right] dt \\
&= h(x) \int_0^\infty E_h^x \left[t < \tau_D : q(X_t) E_h^{X_t} \left[e^{\int_0^{\tau_D} q(X_s)ds} \right] \right] dt
\end{aligned}$$

(by the Markov property for the conditional Brownian motion)

$$\begin{aligned}
&= \int_0^\infty E^x \left[t < \tau_D ; q(X_t) h(X_t) E_h^{X_t} \left[e^{\int_0^{\tau_D} q(X_s)ds} \right] \right] dt \\
&= \int_0^\infty E^x [t < \tau_D ; q(X_t) w(X_t)] dt \\
&= E^x \left[\int_0^{\tau_D} q(X_t) w(X_t) dt \right] \\
&= \int_D G_D(x,y) q(y) w(y) dy.
\end{aligned}$$

$\square$

4. Proof of the Main Theorem

Let $D^\infty = \bar{D} \cup \{\infty\}$ be the compactification of D. Let

$$\mathcal{P}(D) = \{\psi \in C(D) : \lim_{x \to z \in \partial D} \psi(x) \text{ and } \lim_{|x| \to \infty} \psi(x) \quad \text{exist and are finite}\}. \tag{4.1}$$

Obviously $\mathcal{P}(D)$ is isometric to $C(D^\infty)$. $\mathcal{P}(D)$ is a Banach space, with the norm

$$\|\psi\| = \sup_{x \in D} |\psi(x)|.$$

Let q_0 be a positive function belonging to K_2^∞, and let

$$Q_0 = \{q \in K_2^\infty : |q(x)| \le q_0(x), \forall x \in D\}. \tag{4.2}$$

Proposition 4.1. *For a fixed $q_0 > 0$ and $q_0 \in K_2^\infty$, and the harmonic function h as defined in Proposition 2.1, the family of the functions*

$$G_D^h[Q_0] = \left\{ \frac{1}{h(\cdot)} \int_D G_D(\cdot, y) h(y) q(y) dy : q \in Q_0 \right\}$$

is uniformly bounded and equicontinuous in D, and consequently it is relatively compact in $\mathcal{P}(D)$.

Remark. In order to imply the relative compactness of the family in the supremum norm, the equicontinuity of the family in $\mathcal{P}(D)$ should include that the two limits in (4.1) converge uniformly for all functions in the family. This is a modification of the classical Ascoli-Arzelà theorem.

Proof. The uniform integrability of $\left\{\frac{1}{h(x)}G(x,\cdot)h(\cdot)|q(\cdot)| : x \in D\right\}$, by Proposition 3.2, justifies the interchange of the limit and integration, hence we obtain that the function

$$L_q(x) \equiv \frac{1}{h(x)} \int\limits_D G_D(x,y)h(y)q(y)dy$$

is continuous. $\lim_{|x|\to\infty} L_q(x)$ and $\lim_{x\to z\in\partial D} L_q(x)$ exist and are finite. Thus, L_q belongs to $\mathcal{P}(D)$. For any $q \in Q_0$, by (4.2) and 3.2 we have

$$\sup_{x\in D} \int\limits_D \frac{G_D(x,y)}{h(x)} h(y)|q(y)|dy \le \sup_{x\in D} \int\limits_D \frac{G_D(x,y)}{h(x)} h(y)q_0(y)dy$$

$$\le c_D(\|q_0\|_D + \|q_0\|_1).$$

Thus, $G_D^h[Q_0]$ is uniformly bounded. By using the uniform integrability of $\{\frac{1}{h(x)}G_D(x,\cdot)h(\cdot)q_0(\cdot)\}$, we have

i) $\forall q \in Q_0$, $\forall z \in \overline{D}$, as $x, x' \to z$:

$$\left| \int\limits_D \frac{G_D(x,y)}{h(x)} h(y)q(y)dy - \int\limits_D \frac{G_D(x',y)}{h(x')} h(y)q(y)dy \right|$$

$$\le \int\limits_D \left| \frac{G_D(x,y)}{h(x)} - \frac{G_D(x',y)}{h(x')} \right| h(y)|q(y)|dy \tag{4.3}$$

$$\le \int\limits_D \left| \frac{G_D(x,y)}{h(x)} - \frac{G_D(x',y)}{h(x')} \right| h(y)q_0(y)dy \to 0,$$

since $x - x' \to 0$.

ii) $\forall q \in Q_0$,

$$\int\limits_D \frac{G_D(x,y)}{h(x)} h(y)|q(y)|dy \le \int\limits_D \frac{G_D(x,y)}{h(x)} h(y)q_0(y)dy \to 0$$

as $|x| \to \infty$.

Thus, $G_D^h[Q_0]$ is equicontinuous in $\mathcal{P}(D)$, and the limits in (4.1) converge uniformly for all functions in $G_D^h[Q_0]$. Then by the Ascoli-Arzelà theorem $G_D^h[Q_0]$ is relatively compact in $\mathcal{P}(D)$. □

Proof of the Main Theorem. Let $b > 0$ be the real number determined later and, for $c \in (0, b]$, let

$$\Lambda = \left\{ v \in \mathcal{P}(D) : ce^{-\frac{1}{2}} \le v(x) \le 2c, \quad \forall x \in D \right\}. \tag{4.4}$$

For each $v \in \Lambda$ define

$$Tv(x) = cE_h^x \left[e^{\int_0^{\tau_D} q_v(X_t)dt} \right], \quad \forall x \in D, \tag{4.5}$$

where

$$q_v(x) = \frac{f(x, v(x)h(x))}{2v(x)h(x)}. \tag{4.6}$$

For each $x \in D$, by using condition (1.3), (1.4) and the mean value theorem we showed that

$$\left| \frac{f(x, v(x)h(x))}{2v(x)h(x)} \right| \leq F_s(x, 2c\eta \ln(|x| + 1)). \tag{4.7}$$

We now have

$$E_h^x \left[\int_0^{\tau_D} |q_v(X_t)| dt \right] = \frac{1}{h(x)} E^x \left[\int_0^{\tau_D} \frac{h(X_t)|f(X_t, v(X_t)h(X_t))|}{2v(X_t)h(X_t)} dt \right]$$

$$= \frac{1}{h(x)} \int_D \frac{G_D(x,y)h(y)|f(y, v(y)h(y))|}{2v(y)h(y)} dy$$

(by (4.7) and the 3-G Theorem)

$$\leq \mathbf{C} \int_D (G_D(x,y) + h(y) + 1)\, F_s(y, 2c\eta \ln(|y| + 1)) dy. \tag{4.8}$$

Since $F_s(y, \ln(|y| + 1)) \in K_2^\infty \subseteq K_2$, there exists $\alpha \in (0,1)$ such that

$$\sup_{x \in D} \frac{1}{\pi} \int_{|y-x|<\alpha} \ln \frac{1}{|y-x|} F_s(y, \ln(|y|+1)) dy < \frac{1}{4\mathbf{C}}. \tag{4.9}$$

By using $F_s(y, \ln(|y| + 1)) \in K_2^\infty$ again, we have by Proposition 2.3

$$\left\{ h(y) + 1 + \frac{1}{\pi} \left[C_r + \ln \frac{1}{\alpha} + 2 \ln |y - a| \right] \right\} F_s(y, \ln(|y| + 1)) \in L^1(D).$$

Thus, using the monotone convergence theorem and noting that $F_s(y, 0) = 0$ and that $F_s(y, \cdot)$ is a nondecreasing function, there exists $\gamma \in (0, 1)$ such that

$$\int_D \left[h(y) + 1 + \frac{1}{\pi} \left(C_r + \ln \frac{1}{\alpha} + 2 \ln |y - a| \right) \right] F_s(y, \gamma \ln(|y| + 1)) dy < \frac{1}{4\mathbf{C}}. \tag{4.10}$$

Now, we determine b by letting

$$b = \frac{\gamma}{2\eta}. \tag{4.11}$$

Thus, $\forall x \in D$, for $c \in (0, b]$, $2\eta c \leq \gamma < 1$, we have

(4.12)

$$\int_D G_D(x,y) F_s(y, 2\eta c \ln(|y|+1)) dy \leq \int_D G_{B_r^*(a)}(x,y) F_s(y, \gamma \ln(|y|+1)) dy$$

$$\leq \frac{1}{\pi} \int_D (C_r + 2 \ln |y-a| + \ln |x-y|^{-1}) F_s(y, \gamma \ln(|y|+1)) dy$$

by (4.9)

$$\leq \frac{1}{4\mathbf{C}} + \frac{1}{\pi} \int_D \left(C_r + 2 \ln |y - a| + \ln \frac{1}{\alpha} \right) F_s(y, \gamma \ln(|y| + 1)) dy.$$

It follows from (4.8), (4.10), and (4.12) that

$$\sup_{x\in D} E_h^x\left[\int_0^{\tau_D} |q_v(X_t)|dt\right] \le \frac{1}{2}. \tag{4.13}$$

Similarly,

$$\sup_{x\in D} E_h^x\left[\int_0^{\tau_D} F_s(X_t, 2c\eta\ln(|x_t|+1))\right] \le \frac{1}{2}. \tag{4.14}$$

Thus by (4.5), (4.13) and Proposition 3.3 we have

$$ce^{-\frac{1}{2}} \le Tv(x) \le 2c, \tag{4.15}$$

and

$$Tv(x) = c + \frac{1}{h(x)}\int\limits_D G_D(x,y)h(y)q_v(y)Tv(y)dy. \tag{4.16}$$

By (4.15) and the uniform integrability of $\left\{\frac{1}{h(x)}G_D(x,\cdot)h(\cdot)q(\cdot) : x \in D\right\}$, we see that $\lim_{x\to z\in\partial D} Tv(x)$ and $\lim_{|x|\to\infty} Tv(x)$ exist and are finite. Tv is continuous in D. Thus, $Tv \in \mathcal{P}(D)$ and $Tv \in \Lambda$, so

$$T\Lambda \subseteq \Lambda. \tag{4.17}$$

For any $v \in \Lambda$, by (4.7):

$$|q_v(y)Tv(y)| \le 2cF_s(y, \ln(|y|+1)).$$

Let $q_0(y) = 2cF_s(y, \ln(|y|+1))$ and let Q_0 be given as in (4.1). It follows that

$$q_v(\cdot)Tv(\cdot) \in Q_0, \tag{4.18}$$

and

$$\left\{\frac{1}{h(\cdot)}\int\limits_D G_D(\cdot,y)h(y)q_v(y)Tv(y)dy : v \in \Lambda\right\} \subseteq G_D^h[Q_0].$$

We thus have by Proposition 4.1 that $T\Lambda$ is a relatively compact set in $\mathcal{P}(D)$. We shall prove the continuity of T in Λ in the supremum norm. Let $v_n \to v$ in Λ as $n \to \infty$. Since $\tau < \infty$ for P^x-almost Brownian paths, we have by the continuity of function f and the bounded convergence theorem that for almost every Brownian path:

$$\int_0^{\tau_D} \frac{f(X_t, v_n(X_t)h(X_t))}{2v_n(X_t)h(X_t)}dt \to \int_0^{\tau_D} \frac{f(X_t, v(X_t)h(X_t))}{2v(X_t)h(X_t)}dt. \tag{4.19}$$

Since $\{v_n\} \subset \Lambda$, each term of the sequence in (4.19) is bounded by

$$\int_0^{\tau_D} F_s(X_t, 2\eta c\ln(|X_t|+1))dt.$$

By (4.14) and Khas'minskii's lemma

$$E_h^x\left[e^{\int_0^{\tau_D} F_s(X_t, 2\eta c\ln(|X_t|+1))dt}\right] \le 2 < \infty, \tag{4.20}$$

and hence it follows from (4.19), (4.20), and the dominated convergence theorem that $\forall x \in D$

$$E_h^x\left[e^{\int_0^{\tau_D} q_{v_n}(X_t)dt}\right] \to E_h^x\left[e^{\int_0^{\tau_D} q_v(X_t)dt}\right].$$

Thus by (4.5)

$$Tv_n(x) \to Tv(x) \tag{4.21}$$

as $n \to \infty$. We have proved that $T\Lambda$ is a relatively compact family, therefore the pointwise convergence in (4.21) implies the uniform convergence, namely, $\|Tv_n - Tv\| \to 0$ as $n \to \infty$. Thus, we proved that T is a compact and continuous mapping from Λ to itself. By the definition of Λ, Λ is obviously a nonempty, closed, bounded, and convex set in $\mathcal{P}(D)$. Hence by the Schauder fixed-point theorem, there exists a function $v \in \Lambda$ such that

$$Tv = v. \tag{4.22}$$

It follows from (4.16) and (4.22) that

$$v(x) = c + \frac{1}{h(x)} \int_D G_D(x,y)h(y)q_v(y)v(y)dy. \tag{4.23}$$

Now let

$$u(x) = h(x)v(x). \tag{4.24}$$

Since $v \geq ce^{-\frac{1}{2}}$ and $h > 0$ in D, $u > 0$ in D. By (4.6), (4.23), and (4.24), we have

$$u(x) = ch(x) + \frac{1}{2} \int_D G_D(x,y)f(y,u(y))dy. \tag{4.25}$$

Since h is harmonic in D, applying Δ to both sides of (4.25) and using $\Delta(G_D f) = -2f$, we have

$$\begin{aligned} \Delta u(x) &= \frac{1}{2}\Delta \left[\int_D G_D(x,y)f(y,u(y))dy \right] \\ &= -f(x,u(x)). \end{aligned} \tag{4.26}$$

By the property of h and the boundedness of v, we have

$$\lim_{x \to \partial D} u(x) = 0 \tag{4.27}$$

and

$$\lim_{|x| \to \infty} \frac{u(x)}{\ln |x|} = \lim_{|x| \to \infty} \frac{v(x)h(x)}{\ln |x|} = c. \tag{4.28}$$

We thus complete the proof. ∎

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