

EIGENVALUES OF A STURM-LIOUVILLE PROBLEM AND INEQUALITIES OF LYAPUNOV TYPE

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ABSTRACT. We consider the eigenvalue problem $u'' + \lambda u + p(x)u = 0$ in $(0, \pi)$, $u(0) = u(\pi) = 0$, where $p \in L^1(0, \pi)$ keeps a fixed sign and $\|p\|_{L^1} > 0$, and we obtain some lower and upper bounds for $\|p\|_{L^1}$ in terms of its nonnegative eigenvalues λ . Two typical results are: (1) $\|p\|_{L^1} > \sqrt{\lambda} |\sin \sqrt{\lambda} \pi|$ if $\lambda > 1$ and is not the square of a positive integer; (2) $\|p\|_{L^1} \leq 16/\pi$ if $\lambda = 0$ is the smallest eigenvalue.

1. INTRODUCTION

We consider the eigenvalue problem

$$(1) \quad u'' + \lambda u + p(x)u = 0 \quad \text{in } (0, \pi), \quad u(0) = u(\pi) = 0,$$

where $p \in L^1(0, \pi)$. It follows from the classical inequality of Lyapunov that if $\lambda = 0$ is an eigenvalue of the problem (1), then necessarily

$$(2) \quad \|p\|_{L^1} > \pi/4.$$

Over the years there have appeared a number of improvements and extensions of this interesting result (see, e.g. [1], [2]). The purpose of this paper is to extend the original Lyapunov inequality in yet another direction. We treat the problem when λ is any nonnegative eigenvalue of (1).

We assume throughout that

$$(3) \quad p \in L^1(0, \pi), \quad \|p\|_{L^1} > 0 \quad \text{and either } p \geq 0 \text{ or } p \leq 0 \text{ a.e. on } (0, \pi).$$

Our main results are given in §2 in which some lower bounds for $\|p\|_{L^1}$ are obtained in terms of the nonnegative eigenvalues of (1). In §3 we derive some upper bounds for $\|p\|_{L^1}$ which ensure an eigenvalue $\lambda \in [0, 1]$ of (1) to be the smallest eigenvalue. For their proofs, we use basic properties of some boundary value problems and their Green functions (see, e.g. [7]). It was observed in [5] that (2) could be proved using the Green function method. We refer to [6, Chap. 2] for other interesting results relating to the eigenvalue problem (1). This work is motivated by a study on the solvability of some nonlinear boundary value problems (see [3], [4]).

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2. INEQUALITIES OF LYAPUNOV TYPE

In this section some inequalities of Lyapunov type are obtained. We always denote by n a generic positive integer and consider the cases $\lambda \neq n^2$ and $\lambda = n^2$ separately in Theorems 1 and 2 below.

We suppose first that $\lambda \neq n^2$. We recall that for any $h \in L^1(0, \pi)$, the boundary value problem

$$(4) \quad u'' + \lambda u + h(x) = 0 \quad \text{in } (0, \pi), \quad u(0) = u(\pi) = 0$$

has a unique solution u . More precisely, if $\lambda > 0$, we have

$$u(x) = \int_0^\pi G(x, \xi, \lambda)h(\xi) d\xi,$$

where

$$G(x, \xi, \lambda) = \begin{cases} \frac{1}{\sqrt{\lambda} \sin \sqrt{\lambda} \pi} \sin \sqrt{\lambda} \xi \sin \sqrt{\lambda} (\pi - x) & \text{if } 0 \leq \xi \leq x, \\ \frac{1}{\sqrt{\lambda} \sin \sqrt{\lambda} \pi} \sin \sqrt{\lambda} x \sin \sqrt{\lambda} (\pi - \xi) & \text{if } x \leq \xi \leq \pi \end{cases}$$

is the Green function for the problem (4). It follows by a simple calculation that

$$(5) \quad |G(x, \xi, \lambda)| < \begin{cases} \frac{1}{2\sqrt{\lambda}} \tan \frac{\sqrt{\lambda}}{2} \pi & \text{if } 0 < \lambda < 1, \\ \frac{1}{\sqrt{\lambda}} |\csc \sqrt{\lambda} \pi| & \text{if } \lambda > 1 \end{cases}$$

holds a.e. on the square $0 \leq \xi, x \leq \pi$.

Theorem 1. *If $\lambda > 0$ is an eigenvalue of (1), $\lambda \neq n^2$, then*

$$(6) \quad \|p\|_{L^1} > \begin{cases} 2\sqrt{\lambda} \cot \frac{\sqrt{\lambda}}{2} \pi & \text{if } 0 < \lambda < 1, \\ \sqrt{\lambda} |\sin \sqrt{\lambda} \pi| & \text{if } \lambda > 1. \end{cases}$$

Proof. We assume that $p \geq 0$ a.e. on $(0, \pi)$. The other case can be proved similarly. Let u be an eigenfunction of (1) corresponding to λ . Since u is a nontrivial solution to (4) with $h(x) = p(x)u$, it follows from the remark above that

$$(7) \quad u(x) = \int_0^\pi G(x, \xi, \lambda)p(\xi)u(\xi) d\xi.$$

Taking the inner product in $L^2(0, \pi)$ of (7) with $p(x)u$, we have

$$\begin{aligned} \int_0^\pi p(x)|u(x)|^2 dx &= \int_0^\pi \int_0^\pi G(x, \xi, \lambda)p(\xi)u(\xi)p(x)u(x) d\xi dx \\ &< m(\lambda)^{-1} \left(\int_0^\pi p(x)|u(x)| dx \right)^2, \end{aligned}$$

where $m(\lambda)$ is the function defined by the right-hand side of (6), which is reciprocal to that of the right-hand side of (5). Hence

$$(8) \quad \begin{aligned} \int_0^\pi p(x)|u(x)| dx &\leq \left(\int_0^\pi p(x)|u(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_0^\pi p(x) dx \right)^{\frac{1}{2}} \\ &< m(\lambda)^{-\frac{1}{2}} \left(\int_0^\pi p(x)|u(x)| dx \right) \left(\int_0^\pi p(x) dx \right)^{\frac{1}{2}}. \end{aligned}$$

By hypothesis $\int_0^\pi p(x)|u(x)| dx \neq 0$, and so the result follows. □

Obviously the inequality (2) follows from a similar line of arguments using instead the corresponding Green function

$$G(x, \xi, 0) = \begin{cases} \frac{1}{\pi}\xi(\pi - x) & \text{if } 0 \leq \xi \leq x, \\ \frac{1}{\pi}x(\pi - \xi) & \text{if } x \leq \xi \leq \pi. \end{cases}$$

As $G(x, \xi, \lambda)$ is continuous at $\lambda = 0$, we have $\lim_{\lambda \rightarrow 0^+} m(\lambda) = 4/\pi$ as expected.

We now suppose that $\lambda = n^2$. We recall that for any $h \in L^1(0, \pi)$, the boundary value problem

$$(9) \quad u'' + n^2u + h(x) = 0 \quad \text{in } (0, \pi), \quad u(0) = u(\pi) = 0$$

has a solution if and only if $\int_0^\pi h(x) \sin nx \, dx = 0$. If this is the case, there exists a unique solution u to (9) such that $\int_0^\pi u(x) \sin nx \, dx = 0$. More precisely, we have

$$u(x) = \int_0^\pi G(x, \xi, n^2)h(\xi) \, d\xi,$$

where

$$G(x, \xi, n^2) = -\xi \cos n\xi(\sin nx)/n\pi + \begin{cases} \sin n\xi(\cos nx)/n & \text{if } 0 \leq \xi \leq x, \\ \cos n\xi(\sin nx)/n & \text{if } x \leq \xi \leq \pi \end{cases}$$

is the Green function for the problem (9).

Theorem 2. *If $\lambda = n^2$ is an eigenvalue of (1), then*

$$\|p\|_{L^1} > 2n.$$

Proof. We assume that $p \geq 0$ a.e. on $(0, \pi)$. The other case can be proved similarly. Let u be an eigenvalue of (1) corresponding to $\lambda = n^2$. Since u is a nontrivial solution to (9) with $h(x) = p(x)u$, it follows from the remark above that

$$(10) \quad u(x) = (2/\pi) \left(\int_0^\pi u(\xi) \sin n\xi \, d\xi \right) \sin nx + \int_0^\pi G(x, \xi, n^2)p(\xi)u(\xi) \, d\xi.$$

Moreover,

$$(11) \quad \int_0^\pi p(x)u(x) \sin nx \, dx = 0$$

and so taking the inner product in $L^2(0, \pi)$ of (10) with $p(x)u$, we have

$$\begin{aligned} \int_0^\pi p(x)|u(x)|^2 \, dx &= \int_0^\pi \int_0^\pi G(x, \xi, n^2)p(\xi)u(\xi)p(x)u(x) \, d\xi dx \\ &= (1/n) \left[\int_0^\pi \int_0^x \sin n\xi(\cos nx)p(\xi)u(\xi)p(x)u(x) \, d\xi dx \right. \\ &\quad \left. + \int_0^\pi \int_x^\pi \cos n\xi(\sin nx)p(\xi)u(\xi)p(x)u(x) \, d\xi dx \right]. \end{aligned}$$

It follows by (11) again that

$$\begin{aligned} &\left[\int_0^\pi \int_0^x + \int_0^\pi \int_x^\pi \cos n\xi(\sin nx)p(\xi)u(\xi)p(x)u(x) \, d\xi dx \right] \\ &= \int_0^\pi \int_0^\pi \cos n\xi(\sin nx)p(\xi)u(\xi)p(x)u(x) \, d\xi dx = 0 \end{aligned}$$

and hence

$$\begin{aligned} \int_0^\pi p(x)|u(x)|^2 dx &= (1/n) \int_0^\pi \int_0^x \sin n(\xi - x)p(\xi)u(\xi)p(x)u(x) d\xi dx \\ &< (1/2n) \left(\int_0^\pi p(x)|u(x)| dx \right)^2. \end{aligned}$$

We obtain as in (8) that

$$\int_0^\pi p(x)|u(x)| dx < (1/\sqrt{2n}) \left(\int_0^\pi p(x)|u(x)| dx \right) \left(\int_0^\pi p(x) dx \right)^{\frac{1}{2}}.$$

By hypothesis $\int_0^\pi p(x)|u(x)| dx \neq 0$, and so the result follows. \square

We note that the assumption $\|p\|_{L^1} > 0$ is essential for Theorem 2 to hold. Clearly $u = \sin nx$ is an eigenfunction of (1) corresponding to n^2 when $p = 0$ a.e. on $(0, \pi)$.

3. UPPER BOUNDS FOR $\|p\|_{L^1}$

In this section we obtain some upper bounds of $\|p\|_{L^1}$ which are sufficient for an eigenvalue $\lambda \in [0, 1]$ of (1) to be the smallest eigenvalue. We recall that if λ is the smallest eigenvalue of (1), the eigenfunction u corresponding to λ has no zero in $(0, \pi)$. This is the property our proof is based on.

Theorem 3. *Let $0 \leq \lambda \leq 1$. If λ is an eigenvalue of (1) and*

$$(12) \quad \|p\|_{L^1} \leq \begin{cases} \frac{16}{\pi} & \text{if } \lambda = 0, \\ 4\sqrt{\lambda} \cot \frac{\sqrt{\lambda}}{4}\pi & \text{if } 0 < \lambda \leq 1, \end{cases}$$

then λ is the smallest eigenvalue.

Proof. Let u be an eigenfunction of (1) corresponding to λ . We suppose on the contrary that u has a zero $l \in (0, \pi)$ and define

$$v(x) = u\left(\frac{l}{\pi}x\right), \quad w(x) = u\left(\frac{\pi-l}{\pi}x + l\right).$$

Then v and w are nontrivial solutions of (1) with λ replaced by $(\frac{l}{\pi})^2\lambda$ and $(\frac{\pi-l}{\pi})^2\lambda$, $p(x)$ replaced by $(\frac{l}{\pi})^2p(\frac{l}{\pi}x)$ and $(\frac{\pi-l}{\pi})^2p(\frac{\pi-l}{\pi}x + l)$, respectively.

If $0 < \lambda \leq 1$, we apply Theorem 1 to obtain

$$\int_0^l |p(x)| dx > 2\sqrt{\lambda} \cot \frac{l}{2}\sqrt{\lambda}; \quad \int_l^\pi |p(x)| dx > 2\sqrt{\lambda} \cot \frac{\pi-l}{2}\sqrt{\lambda}.$$

It follows by a simple calculation that

$$\begin{aligned} \int_0^\pi |p(x)| dx &> 2\sqrt{\lambda} \left[\cot \frac{l}{2}\sqrt{\lambda} + \cot \frac{\pi-l}{2}\sqrt{\lambda} \right] \\ &\geq 4\sqrt{\lambda} \cot \frac{\sqrt{\lambda}}{4}\pi \end{aligned}$$

which contradicts (12).

If $\lambda = 0$, we apply (2) to obtain

$$\begin{aligned}\int_0^\pi |p(x)| dx &= \left[\int_0^l + \int_l^\pi |p(x)| dx \right] \\ &> 4 \left(\frac{1}{l} + \frac{1}{\pi - l} \right) \geq \frac{16}{\pi}\end{aligned}$$

which again contradicts (12). This completes the proof of the theorem. \square

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