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WICK PRODUCTS OF THE CAR ALGEBRA

E. R. NEGRIN

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Abstract. The purpose of this paper is to put in a precise mathematical (algebraic) form the Wick products of the CAR algebra. We state in detail the reduction of the ordinary product of Fermi fields in terms of a finite sum of monomials in the creation and annihilation operators in which all creation operators occur to the left of all annihilation operators (Wick-ordered) and the Fock (vacuum) state of the former.

1. Introduction

Let $\mathbf{H}$ be a complex Hilbert space, and let $\mathcal{C}(\mathbf{H})$ be the CAR (canonical anticommutation relation) algebra over $\mathbf{H}$, i.e., the C^*-algebra generated by the set

$$\{a^*(f), a(f) \mid f \in \mathbf{H}\},$$

together with a unit e, and whose elements satisfy the anticommutation relations

$$a(f)a^*(g) + a^*(g)a(f) = \langle f, g\rangle e, \tag{1.1}$$

$$a^*(f)a^*(g) + a^*(g)a^*(f) = 0, \tag{1.2}$$

$$a(f)a(g) + a(g)a(f) = 0, \quad f, g \in \mathbf{H}. \tag{1.3}$$

We will denote

$$\psi(f) = a^*(f) + a(f), \quad f \in \mathbf{H}.$$

As usual, we will refer to $\psi(f)$ as the Fermi field, and $a^*(f)$ and $a(f)$ as the creation and annihilation operators, respectively.

Here, we obtain an explicit expression for the reduction of the product $\psi(f_1)\cdots\psi(f_n)$ in terms of Wick-ordered products, in which all creation operators occur to the left of all annihilation operators, that is, a finite linear combination of monomials of the form $a^*(f_1)\cdots a^*(f_m)a(g_1)\cdots a(g_n)$, for vectors $f_i, g_j \in \mathbf{H}$.

This program, developed first by F.J. Dyson [4] and G.C. Wick [9], has been useful in practical quantum mechanics, especially in applications to Feynman graph theory, where the Fermi field plays an important role in defining Feynman propagators as the Fock (vacuum) state of ordinary products of them (cf. [2, §17.4], [3, §17] and [5, §4-2].)

Our treatment benefits immeasurably from the interpretation on Wick products of Bose fields given by I.E. Segal (cf. [7], [8, §1.4]) and I.E. Segal et al. [1, §7.2].

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2. Algebraic process of reduction

Definition 1. A Fock (vacuum) state on $\mathcal{C}(\mathbf{H})$ is a linear functional E on $\mathcal{C}(\mathbf{H})$ such that $E(e) = 1$ and

$$E\left(a^*(f_1)\cdots a^*(f_m)a(g_1)\cdots a(g_n)\right) = 0,$$

$$E\left(a^*(f_1)\cdots a^*(f_m)\right) = 0,$$

$$E\left(a(g_1)\cdots a(g_n)\right) = 0,$$

for all $f_1, \ldots, f_m$ and $g_1, \ldots, g_n$ in $\mathbf{H}$, and all $m, n \in \mathbb{N}$.

Note that, from [6, formula (2.5), p. 252] there exists a unique linear functional satisfying the preceding conditions.

An explicit mathematical definition of Wick-ordered products of the CAR algebra may be formulated as follows.

Definition 2. The Wick product of $\psi(f_1)\cdots\psi(f_n)$, with $f_1, \ldots, f_n$ in $\mathbf{H}$, is the element of $\mathcal{C}(\mathbf{H})$,

(2.1)
$$:\psi(f_1)\cdots\psi(f_n): = \sum_{S\subseteq\{1,\ldots,n\}} \delta_P \left\{\prod_{k\in S} a^*(f_k)\right\}\left\{\prod_{k\in\{1,\ldots,n\}-S} a(f_k)\right\},$$

where δ_P denotes the sign of the permutation needed to permute $1, \ldots, n$ into the order in which it appears in the formula, and the summation ranges over all subsets S of $\{1, \ldots, n\}$.

Note that from relations (1.1), (1.2) and (1.3), the products involved in formula (2.1) are independent of the order of the factors. Thus, the Wick product $:\psi(f_1)\cdots\psi(f_n):$ is well defined.

The next result will be useful.

Lemma. *For any $f_1, \ldots, f_n$ in $\mathbf{H}$, $n \in \mathbb{N}$, $n > 1$, it follows that*

$$:\psi(f_1)\cdots\psi(f_{n-1}):\psi(f_n) = :\psi(f_1)\cdots\psi(f_{n-1})\psi(f_n):$$

(2.2)
$$+ \sum_{1\le i\le n-1} \delta_i :\psi(f_1)\cdots\widehat{\psi(f_i)}\cdots\psi(f_{n-1}): E(\psi(f_i)\psi(f_n)),$$

where δ_i denotes the sign of the permutation $1, \ldots, \widehat{i}, \ldots, n-1, i, n$.

Proof. We will prove this Lemma by induction on n. For $n = 2$, it establishes that $\psi(f_1)\psi(f_2) = :\psi(f_1)\psi(f_2): + E(\psi(f_1)\psi(f_2))$, which is easily seen to be valid by direct computations from (2.1).

Now,

$$
\begin{aligned}
&:\psi(f_1)\cdots\psi(f_{n-1}):\psi(f_n) \\
&\quad= \sum_{S\subseteq\{1,\ldots,n-1\}} \delta_P \left\{\prod_{k\in S} a^*(f_k)\right\} \left\{\prod_{k\in\{1,\ldots,n-1\}-S} a(f_k)\right\} \psi(f_n) \\
&\quad= \sum_{S\subseteq\{1,\ldots,n-1\}} \delta_P \left\{\prod_{k\in S} a^*(f_k)\right\} \left\{\prod_{k\in\{1,\ldots,n-1\}-S} a(f_k)\right\} (a^*(f_n)+a(f_n)) \\
&\quad= \sum_{S\subseteq\{1,\ldots,n-1\}} \delta_P \left\{\prod_{k\in S} a^*(f_k)\right\} \left\{\prod_{k\in\{1,\ldots,n-1\}-S} a(f_k)\right\} a^*(f_n) \\
&\qquad+ \sum_{S\subseteq\{1,\ldots,n-1\}} \delta_P \left\{\prod_{k\in S} a^*(f_k)\right\} \left\{\prod_{k\in\{1,\ldots,n-1\}-S} a(f_k)\right\} a(f_n).
\end{aligned} \tag{2.3}
$$

From relations (1.1), (1.2) and (1.3), the fact that $E(\psi(f_1)\psi(f_2)) = \langle f_2, f_1\rangle$ and the definition of δ_P, formula (2.3) yields (2.2). □

With this result, the reduction is obtained in

Theorem. *Let $\mathcal{C}(\mathbf{H})$ be the CAR algebra over a complex Hilbert space $\mathbf{H}$, and let E be the Fock (vacuum) state on $\mathcal{C}(\mathbf{H})$. Then, for any $f_1,\ldots,f_n$ in $\mathbf{H}$,*

$$
\begin{aligned}
\psi(f_1)\cdots\psi(f_n) &= :\psi(f_1)\cdots\psi(f_n): \\
&+ \sum_{1\le j<k\le n} \delta_{jk} :\psi(f_1)\cdots\widehat{\psi(f_j)}\cdots\widehat{\psi(f_k)}\cdots\psi(f_n): E(\psi(f_j)\psi(f_k)) \\
&+ \sum_{1\le j_1<k_1\le n, 1\le j_2<k_2\le n} \delta_{j_1k_1j_2k_2} :\psi(f_1)\cdots\widehat{\psi(f_{j_1})}\cdots\widehat{\psi(f_{k_1})} \\
&\qquad \cdots\widehat{\psi(f_{j_2})}\cdots\widehat{\psi(f_{k_2})}\cdots\psi(f_n): E(\psi(f_{j_1})\psi(f_{k_1}))\, E(\psi(f_{j_2})\psi(f_{k_2})) \\
&+\cdots+ \sum_{1\le j_1<k_1\le n,\ldots,1\le j_s<k_s\le n} \delta_{j_1k_1\ldots j_sk_s}\, u\, E(\psi(f_{j_1})\psi(f_{k_1}))\cdots E(\psi(f_{j_s})\psi(f_{k_s})),
\end{aligned}
$$

where s denotes the integral part of $n/2$, and if n is even $u = e$ (the identity in $\mathcal{C}(\mathbf{H})$), but if n is odd $u = \psi(f_t)$, where t is the unique integer such that $1 \le t \le n$ and $t \ne j_m$, $t \ne k_m$ for all $m = 1,\ldots,s$.

In both cases, the sums are taken only over j_m, k_m such that $j_m \ne j_n$, $j_m \ne k_n$ and $k_m \ne k_n$ for all $m, n \in \mathbb{N}$ with $m \ne n$. Moreover, the sums range only over $j_1 < j_2 < \cdots < j_m$.

Also, in both cases, $\delta_{i_1\ldots i_m}$ denotes the sign of the permutation

$$1,\ldots,\widehat{i_1},\ldots,\widehat{i_m},\ldots,n,i_1,\ldots,i_m.$$

Proof. We will prove this result by induction on n. Clearly $\psi(f_1) = :\psi(f_1):$. For the case when $n = 2$, one has $\psi(f_1)\psi(f_2) = :\psi(f_1)\psi(f_2): + E(\psi(f_1)\psi(f_2))$, which is easily seen to be valid.

Assume n is odd. By the induction hypothesis

$$\psi(f_1)\cdots\psi(f_{n-1})\psi(f_n) = :\psi(f_1)\cdots\psi(f_{n-1}):\psi(f_n)$$
$$+\sum_{1\le j<k\le n-1}\delta'_{jk}:\psi(f_1)\cdots\widehat{\psi(f_j)}\cdots\widehat{\psi(f_k)}\cdots\psi(f_{n-1}):\psi(f_n)E\left(\psi(f_j)\psi(f_k)\right)$$
$$+\sum_{1\le j_1<k_1\le n-1,1\le j_2<k_2\le n-1}\delta'_{j_1k_1j_2k_2}:\psi(f_1)\cdots\widehat{\psi(f_{j_1})}\cdots\widehat{\psi(f_{k_1})}$$
$$\cdots\widehat{\psi(f_{j_2})}\cdots\widehat{\psi(f_{k_2})}\cdots\psi(f_{n-1}):\psi(f_n)E\left(\psi(f_{j_1})\psi(f_{k_1})\right)E\left(\psi(f_{j_2})\psi(f_{k_2})\right)$$
$$+\cdots+\sum_{1\le j_1<k_1\le n-1,\ldots,1\le j_{s_*}<k_{s_*}\le n-1}\delta'_{j_1k_1\ldots j_{s_*}k_{s_*}}\psi(f_n)E\left(\psi(f_{j_1})\psi(f_{k_1})\right)$$
$$\cdots E\left(\psi(f_{j_{s_*}})\psi(f_{k_{s_*}})\right),$$

where s_* denotes the integral part of $(n-1)/2$, and $\delta'_{i_1\ldots i_m}$ denotes the sign of the permutation

$$1,\ldots,\widehat{i_1},\ldots,\widehat{i_m},\ldots,n-1,i_1,\ldots,i_m.$$

Now, using the preceding Lemma, the equality above becomes

$$\psi(f_1)\cdots\psi(f_{n-1})\psi(f_n) = :\psi(f_1)\cdots\psi(f_{n-1})\psi(f_n):$$
$$+\sum_{1\le i\le n-1}\delta^i_i:\psi(f_1)\cdots\widehat{\psi(f_i)}\cdots\psi(f_{n-1}):E\left(\psi(f_i)\psi(f_n)\right)$$
$$+\sum_{1\le j<k\le n-1}\delta'_{jk}:\psi(f_1)\cdots\widehat{\psi(f_j)}\cdots\widehat{\psi(f_k)}\cdots\psi(f_n):E\left(\psi(f_j)\psi(f_k)\right)$$
$$+\sum_{1\le j<k\le n-1}\ \sum_{1\le i\le n-1,i\ne j,i\ne k}\delta^{jk}_i\delta'_{jk}:\psi(f_1)\cdots\widehat{\psi(f_j)}\cdots\widehat{\psi(f_k)}\cdots\widehat{\psi(f_i)}\cdots\psi(f_{n-1}):$$
$$E\left(\psi(f_j)\psi(f_k)\right)E\left(\psi(f_i)\psi(f_n)\right)$$
$$+\sum_{1\le j_1<k_1\le n-1,1\le j_2<k_2\le n-1}\delta'_{j_1k_1j_2k_2}:\psi(f_1)\cdots\widehat{\psi(f_{j_1})}\cdots\widehat{\psi(f_{k_1})}$$
$$\cdots\widehat{\psi(f_{j_2})}\cdots\widehat{\psi(f_{k_2})}\psi(f_{n-1})\psi(f_n):E\left(\psi(f_{j_1})\psi(f_{k_1})\right)E\left(\psi(f_{j_2})\psi(f_{k_2})\right)$$
$$+\sum_{\substack{1\le j_1<k_1\le n-1\\1\le j_2<k_2\le n-1}}\ \sum_{\substack{1\le i\le n-1\\i\ne j_1,i\ne j_2,i\ne k_2}}\delta^{j_1k_1j_2k_2}_i\delta'_{j_1k_1j_2k_2}:\psi(f_1)\cdots\widehat{\psi(f_{j_1})}$$
$$\cdots\widehat{\psi(f_{k_1})}\cdots\widehat{\psi(f_{j_2})}\cdots\widehat{\psi(f_{k_2})}\cdots\widehat{\psi(f_i)}\cdots\psi(f_{n-1}):$$
$$E\left(\psi(f_{j_1})\psi(f_{k_1})\right)E\left(\psi(f_{j_2})\psi(f_{k_2})\right)E\left(\psi(f_i)\psi(f_n)\right)$$
$$+\cdots+\sum_{\substack{1\le j_1<k_1\le n-1,\ldots\\1\le j_{s_*-1}<k_{s_*-1}\le n-1}}\ \sum_{\substack{1\le i\le n,i\ne j_1\\i\ne k_1,\ldots,i\ne j_{s_*-1},i\ne k_{s_*-1}}}\delta^{j_1k_1\ldots j_{s_*-1}k_{s_*-1}}_i$$
$$\delta'_{j_1k_1\ldots j_{s_*-1}k_{s_*-1}}\psi(f_t)E\left(\psi(f_{j_1})\psi(f_{k_1})\right)\cdots E\left(\psi(f_{j_{s_*-1}})\psi(f_{k_{s_*-1}})\right)E\left(\psi(f_i)\psi(f_n)\right)$$
$$+\sum_{1\le j_1<k_1\le n-1,\ldots,1\le j_{s_*}<k_{s_*}\le n-1}\delta'_{j_1k_1\ldots j_{s_*}k_{s_*}}\psi(f_n)E\left(\psi(f_{j_1})\psi(f_{k_1})\right)\tag{2.4}$$
$$\cdots E\left(\psi(f_{j_{s_*}})\psi(f_{k_{s_*}})\right),$$

where t is the unique integer such that $1 \leq t \leq n-1$ and $t \neq i$, $t \neq j_m$, $t \neq k_m$, $i \neq j_m$, $i \neq k_m$, for all $m = 1, \ldots, s_* - 1$ and $1 \leq i \leq n-1$. Moreover, $\delta_i^{i_1 \ldots i_m}$ denotes the sign of the permutation

$$1, \ldots, \widehat{i_1}, \ldots, \widehat{i_m}, \ldots, \widehat{i}, \ldots, n-1, i, n\,.$$

With regard to the sums of each two consecutive terms in formula (2.4) one has

$$\begin{aligned}
&\sum_{1 \leq j_1 < k_1 \leq n-1, \ldots, 1 \leq j_m < k_m \leq n-1} \; \sum_{1 \leq i \leq n-1, i \neq j_1, i \neq k_1, \ldots, i \neq j_m, i \neq k_m} \delta_i^{j_1 k_1 \ldots j_m k_m} \delta'_{j_1 k_1 \ldots j_m k_m} \\
&\quad :\psi(f_1) \cdots \widehat{\psi(f_{j_1})} \cdots \widehat{\psi(f_{k_1})} \cdots \widehat{\psi(f_{j_m})} \cdots \widehat{\psi(f_{k_m})} \cdots \widehat{\psi(f_i)} \cdots \psi(f_{n-1}): \\
&\quad E\left(\psi(f_{j_1})\psi(f_{k_1})\right) \cdots E\left(\psi(f_{j_m})\psi(f_{k_m})\right) E\left(\psi(f_i)\psi(f_n)\right) \\
&+ \sum_{1 \leq j_1 < k_1 \leq n-1, \ldots, 1 \leq j_{m+1} < k_{m+1} \leq n-1} \delta'_{j_1 k_1 \ldots j_{m+1} k_{m+1}} :\psi(f_1) \cdots \widehat{\psi(f_{j_1})} \cdots \widehat{\psi(f_{k_1})} \\
&\quad \cdots \widehat{\psi(f_{j_{m+1}})} \cdots \widehat{\psi(f_{k_{m+1}})} \cdots \psi(f_n): E\left(\psi(f_{j_1})\psi(f_{k_1})\right) \cdots E\left(\psi(f_{j_{m+1}})\psi(f_{k_{m+1}})\right) \\
&= \sum_{1 \leq j_1 < k_1 \leq n, \ldots, 1 \leq j_m < k_m \leq n} \delta_{j_1 k_1 \ldots j_m k_m} :\psi(f_1) \cdots \widehat{\psi(f_{j_1})} \cdots \widehat{\psi(f_{k_1})} \cdots \widehat{\psi(f_{j_m})} \\
&\quad \cdots \widehat{\psi(f_{k_m})} \cdots \psi(f_n): E\left(\psi(f_{j_1})\psi(f_{k_1})\right) \cdots E\left(\psi(f_{j_m})\psi(f_{k_m})\right).
\end{aligned}$$

Using this fact, one obtains the desired result for n odd.

Now, assume n is even. By the induction hypothesis,

$$\begin{aligned}
&\psi(f_1) \cdots \psi(f_{n-1})\psi(f_n) = :\psi(f_1) \cdots \psi(f_{n-1}): \psi(f_n) \\
&+ \sum_{1 \leq j < k \leq n-1} \delta'_{jk} :\psi(f_1) \cdots \widehat{\psi(f_j)} \cdots \widehat{\psi(f_k)} \cdots \psi(f_{n-1}): \psi(f_n) E\left(\psi(f_j)\psi(f_k)\right) \\
&+ \sum_{1 \leq j_1 < k_1 \leq n-1, 1 \leq j_2 < k_2 \leq n-1} \delta'_{j_1 k_1 j_2 k_2} :\psi(f_1) \cdots \widehat{\psi(f_{j_1})} \cdots \widehat{\psi(f_{k_1})} \cdots \widehat{\psi(f_{j_2})} \cdots \widehat{\psi(f_{k_2})} \\
&\quad \cdots \psi(f_{n-1}): \psi(f_n) E\left(\psi(f_{j_1})\psi(f_{k_1})\right) E\left(\psi(f_{j_2})\psi(f_{k_2})\right) \\
&+ \cdots + \sum_{1 \leq j_1 < k_1 \leq n-1, \ldots, 1 \leq j_{s_*} < k_{s_*} \leq n-1} \delta'_{j_1 k_1 \ldots j_{s_*} k_{s_*}} \psi(f_t)\psi(f_n) E\left(\psi(f_{j_1})\psi(f_{k_1})\right) \\
&\quad \cdots E\left(\psi(f_{j_{s_*}})\psi(f_{k_{s_*}})\right),
\end{aligned}$$

where s_* denotes the integral part of $(n-1)/2$ and $\delta'_{i_1 \ldots i_m}$ denotes the sign of the permutation

$$1, \ldots, \widehat{i_1}, \ldots, \widehat{i_m}, \ldots, n-1, i_1, \ldots, i_m\,.$$

Moreover t is the unique integer such that $1 \leq t \leq n-1$, and $t \neq j_m$, $t \neq k_m$, for all $m = 1, \ldots, s_*$.

Using again the preceding Lemma, the equality above becomes

$$\begin{aligned}
&\psi(f_1) \cdots \psi(f_{n-1})\psi(f_n) = :\psi(f_1) \cdots \psi(f_{n-1})\psi(f_n): \\
&+ \sum_{1 \leq i \leq n-1} \delta_i^i :\psi(f_1) \cdots \widehat{\psi(f_i)} \cdots \psi(f_{n-1}): E\left(\psi(f_i)\psi(f_n)\right) \\
&+ \sum_{1 \leq j < k \leq n-1} \delta'_{jk} :\psi(f_1) \cdots \widehat{\psi(f_j)} \cdots \widehat{\psi(f_k)} \cdots \psi(f_{n-1})\psi(f_n): E\left(\psi(f_j)\psi(f_k)\right) \\
&+ \sum_{1 \leq j < k \leq n-1} \; \sum_{1 \leq i \leq n-1, i \neq j, i \neq k} \delta_i^{jk} \delta'_{jk} :\psi(f_1) \cdots \widehat{\psi(f_j)} \cdots \widehat{\psi(f_k)} \cdots \widehat{\psi(f_i)} \cdots \psi(f_{n-1}): \\
&\quad E\left(\psi(f_j)\psi(f_k)\right) E\left(\psi(f_i)\psi(f_n)\right)
\end{aligned}$$

$$+\sum_{1\le j_1<k_1\le n-1,1\le j_2<k_2\le n-1}\delta'_{j_1k_1j_2k_2}:\psi(f_1)\cdots\widehat{\psi(f_{j_1})}\cdots\widehat{\psi(f_{k_1})}\cdots\widehat{\psi(f_{j_2})}\cdots\widehat{\psi(f_{k_2})}$$
$$\cdots\psi(f_{n-1})\psi(f_n):E\left(\psi(f_{j_1})\psi(f_{k_1})\right)E\left(\psi(f_{j_2})\psi(f_{k_2})\right)$$
$$+\sum_{1\le j_1<k_1\le n-1,1\le j_2<k_2\le n-1}\ \sum_{1\le i\le n-1,i\ne j_1,i\ne k_1,i\ne j_2,i\ne k_2}\delta_i^{j_1k_1j_2k_2}\delta'_{j_1k_1j_2k_2}$$
$$:\psi(f_1)\cdots\widehat{\psi(f_{j_1})}\cdots\widehat{\psi(f_{k_1})}\cdots\widehat{\psi(f_{j_2})}\cdots\widehat{\psi(f_{k_2})}\cdots\widehat{\psi(f_i)}\cdots\psi(f_{n-1}):$$
$$E\left(\psi(f_{j_1})\psi(f_{k_1})\right)E\left(\psi(f_{j_2})\psi(f_{k_2})\right)E\left(\psi(f_i)\psi(f_n)\right)$$
$$+\cdots+\sum_{1\le j_1<k_1\le n-1,\dots,1\le j_{s_*-1}<k_{s_*-1}\le n-1}\ \sum_{1\le i\le n-1,i\ne j_1,i\ne k_1,\dots,i\ne j_{s_*-1},i\ne k_{s_*-1}}$$
$$\delta_i^{j_1k_1\dots j_{s_*-1}k_{s_*-1}}\delta'_{j_1k_1\dots j_{s_*-1}k_{s_*-1}}:\psi(f_p)\psi(f_q):E\left(\psi(f_{j_1})\psi(f_{k_1})\right)$$
$$\cdots E\left(\psi(f_{j_{s_*-1}})\psi(f_{k_{s_*-1}})\right)E\left(\psi(f_i)\psi(f_n)\right)$$
$$+\sum_{1\le j_1<k_1\le n-1,\dots,1\le j_{s_*}<k_{s_*}\le n-1}\delta'_{j_1k_1\dots j_{s_*}k_{s_*}}$$
$$\psi(f_t)\psi(f_n)E\left(\psi(f_{j_1})\psi(f_{k_1})\right)\cdots E\left(\psi(f_{j_{s_*}})\psi(f_{k_{s_*}})\right), \tag{2.5}$$

where p and q denote the unique integers $1\le p<q\le n-1$, such that $p\ne i$, $q\ne i$, $p\ne j_m$, $q\ne k_m$, $i\ne j_m$, $i\ne k_m$, for all $m=1,\dots s_*-1$ and $1\le i\le n-1$. Moreover, $\delta_i^{i_1\dots i_m}$ denotes the sign of the permutation

$$1,\dots,\widehat{i_1},\dots,\widehat{i_m},\dots,\widehat{i},\dots,n-1,i,n\,.$$

Now, arguing as in the case when n is odd, using the fact that $\psi(f_t)\psi(f_n)=\,:\psi(f_t)\psi(f_n):+E\left(\psi(f_t)\psi(f_n)\right)$ into formula (2.5), and taking into account the sign of the permutation, one obtains the desired result for n even. □

As a straightforward consequence of this result we have the following

Corollary. *Let $\mathcal{C}(\mathbf{H})$ be the CAR algebra over a complex Hilbert space $\mathbf{H}$, and let E be the Fock (vacuum) state on $\mathcal{C}(\mathbf{H})$. Then, for any $f_1,\dots,f_n$ in $\mathbf{H}$, one has:*
a) If n is odd, $E\left(\psi(f_1)\cdots\psi(f_n)\right)=0$.
b) If n is even,

$$E\left(\psi(f_1)\cdots\psi(f_n)\right)$$
$$=\sum_{1\le j_1<k_1\le n,\dots,1\le j_s<k_s\le n}\delta_{j_1k_1\dots j_sk_s}E\left(\psi(f_{j_1})\psi(f_{k_1})\right)\cdots E\left(\psi(f_{j_s})\psi(f_{k_s})\right),$$

where the sums, s and δ are defined as in the Theorem above.

Remark. It is clarifying to compute directly from (2.1) the following expressions (which agree with the formulae obtained in the Theorem and Corollary above):

$$\psi(f_1)\psi(f_2)\psi(f_3)=\,:\psi(f_1)\psi(f_2)\psi(f_3):+\psi(f_1)E\left(\psi(f_2)\psi(f_3)\right)$$
$$-\psi(f_2)E\left(\psi(f_1)\psi(f_3)\right)+\psi(f_3)E\left(\psi(f_1)\psi(f_2)\right),$$

and

$$\psi(f_1)\psi(f_2)\psi(f_3)\psi(f_4)=\,:\psi(f_1)\psi(f_2)\psi(f_3)\psi(f_4):+:\psi(f_1)\psi(f_2):E\left(\psi(f_3)\psi(f_4)\right)$$
$$-:\psi(f_1)\psi(f_3):=E\left(\psi(f_2)\psi(f_4)\right)+:\psi(f_1)\psi(f_4):E\left(\psi(f_2)\psi(f_3)\right)$$
$$+:\psi(f_2)\psi(f_3):E\left(\psi(f_1)\psi(f_4)\right)-:\psi(f_2)\psi(f_4):E\left(\psi(f_1)\psi(f_3)\right)$$
$$+:\psi(f_3)\psi(f_4):E\left(\psi(f_1)\psi(f_2)\right)+E\left(\psi(f_1)\psi(f_2)\right)E\left(\psi(f_3)\psi(f_4)\right)$$
$$-E\left(\psi(f_1)\psi(f_3)\right)E\left(\psi(f_2)\psi(f_4)\right)+E\left(\psi(f_1)\psi(f_4)\right)E\left(\psi(f_2)\psi(f_3)\right).$$

For the Fock (vacuum) state values we obtain

$$E\left(\psi(f_1)\psi(f_2)\psi(f_3)\right) = 0,$$

and

$$E\left(\psi(f_1)\psi(f_2)\psi(f_3)\psi(f_4)\right) = E\left(\psi(f_1)\psi(f_2)\right) E\left(\psi(f_3)\psi(f_4)\right)$$
$$-E\left(\psi(f_1)\psi(f_3)\right) E\left(\psi(f_2)\psi(f_4)\right) + E\left(\psi(f_1)\psi(f_4)\right) E\left(\psi(f_2)\psi(f_3)\right).$$

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Departamento de Análisis Matemático, Universidad de La Laguna, 38271 Canary Islands, Spain

E-mail address: enegrin@ull.es