

**A CLASS OF M -DILATION SCALING FUNCTIONS
WITH REGULARITY GROWING PROPORTIONALLY
TO FILTER SUPPORT WIDTH**

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ABSTRACT. In this paper, a class of M -dilation scaling functions with regularity growing proportionally to filter support width is constructed. This answers a question proposed by Daubechies on p.338 of her book *Ten Lectures on Wavelets* (1992).

1. INTRODUCTION

Let $M \geq 2$ be a fixed integer. A multiresolution analysis for dilation M consists of a sequence of closed subspaces V_j of $L^2(\mathbb{R})$ that satisfy the following conditions (see [C], [D], [M]):

- i) $V_j \subset V_{j+1}, \forall j \in \mathbb{Z}$;
- ii) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$;
- iii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;
- iv) $f \in V_j \iff f(2^{-j}\cdot) \in V_0$;
- v) there exists a function ϕ in V_0 such that $\{\phi(\cdot - n); n \in \mathbb{Z}\}$ is an orthonormal basis of V_0 .

The function ϕ is called an M -dilation scaling function. It is easy to see that ϕ satisfies the refinement equation

$$(1) \quad \phi(x) = \sum_{n \in \mathbb{Z}} c_n \phi(Mx - n),$$

where the sequence $\{c_n\}$ satisfies

$$\sum_{n \in \mathbb{Z}} c_n = M.$$

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In this paper we shall only deal with compactly supported M -dilation scaling functions. In this case the sequence $\{c_n\}$ must have finite length. The function

$$(2) \quad H(\xi) = \frac{1}{M} \sum_{n \in \mathbb{Z}} c_n e^{in\xi}$$

is called a symbol corresponding to the refinement equation (1).

The filter support width $W(\phi)$ of an M -dilation scaling function ϕ is defined as the difference of the largest and the smallest indices of the nonzero c_n . The regularity $R(\phi)$ of ϕ is defined as the supremum of α such that $\phi \in C^\alpha$, where C^α denotes the Hölder class of index α .

In her book [D, p.338], Daubechies remarks that:

At present, I know of no explicit scheme that provides an infinite family of m_0 (i.e., symbol H), for dilation 3 (i.e., $M = 3$), with regularity growing proportionally to the filter support width.

To our knowledge, this question is still open. The purpose of this paper is to construct a class of M -dilation scaling functions ϕ_N for which there exists a constant λ_M independent of N such that

$$(3) \quad R(\phi_N) \geq \lambda_M W(\phi_N),$$

where $M \geq 3$. On the other hand it is already known that (see [DL])

$$W(\phi_N) \geq R(\phi_N).$$

These facts give an affirmative answer to Daubechies' question.

The regularity of ϕ has been studied widely; see for instance [CL], [BDS], [D], [HW1], [HW2], [So], [S] and [WL]. In general, to study the regularity of ϕ we need to consider the symbol (2) first. By the Fourier transform, we see that all the symbols H satisfy

$$(4) \quad \sum_{l=0}^{M-1} |H(\xi + \frac{2l\pi}{M})|^2 = 1.$$

The solutions H of the equation (4) are determined by (see [BDS], [H], [HSZ])

$$(5) \quad |H(\xi)|^2 = \left(\frac{\sin^2(M\xi/2)}{M^2 \sin^2 \xi/2}\right)^N \sum_{s=0}^{N-1} {}_N a(s) \sin^{2s} \frac{\xi}{2} + \left(\sin \frac{M\xi}{2}\right)^{2N} R(\xi),$$

where

$${}_N a(s) = \sum_{s_1 + \dots + s_{M-1} = s} \prod_{j=1}^{M-1} \binom{N-1+s_j}{s_j} \frac{1}{\sin^{2s_j} j\pi/M}$$

and R is a real-valued trigonometric polynomial such that $\sum_{l=0}^{M-1} R(\xi + 2l\pi/M) = 0$ and the right hand side of (5) is nonnegative.

By the Riesz Lemma (see [D, p.172]), such symbol H exists. Let ${}_N H$ be a solution of (5) with $R = 0$, and let ${}_N \phi$ be the solution of (1) corresponding to the symbol ${}_N H$. In [BDS], Bi, Dai and Sun prove the following estimates on the regularity of ${}_N \phi$:

$$|R({}_N \phi) - \frac{\ln N}{4 \ln M}| \leq C$$

when M is odd, and

$$|R_{(N\phi)} - \frac{4N \ln \left(\sin \frac{M\pi}{2M+2} \right)^{-1} + \ln N}{4 \ln M}| \leq C$$

when M is even. A more precise estimate of $R_{(N\phi)}$ can be found in [S]. For the special cases $M = 3, 4, 5$, similar results are obtained by Soardi ([So]) and Heller and Wells ([HW2]) independently. This result shows that for these special $N\phi$ the regularity does not grow proportionally to the filter support width when M is odd. To construct M -dilation scaling functions with regularity growing proportionally to the filter support width we use the symbol H_N determined by

$$|H_N(\xi)|^2 = \sum_{k_0 + \dots + k_{M-1} = MN - M + 1} \alpha_N(k_0, \dots, k_{M-1}) \frac{(MN - M + 1)!}{k_0! \dots k_{M-1}!} \times \prod_{l=0}^{M-1} \left(\frac{\sin M\xi/2}{M \sin(\xi/2 + l\pi/M)} \right)^{2k_l},$$

where $N \geq 1$, and $\alpha_N(k_0, \dots, k_{M-1})$ is defined by

$$\alpha_N(k_0, \dots, k_{M-1}) = \begin{cases} 0, & \text{if } k_0 \leq N - 1, \\ \frac{1}{\#(E)}, & \text{if } k_0 \geq N, \end{cases}$$

where $E = \{j : k_j \geq N\}$ and $\#(E)$ is the cardinality of E . Let ϕ_N be the solution of (1) corresponding to a symbol H_N . Then we have the following

Theorem. *Let $M \geq 3$ and $N \geq 2$ be any natural numbers. Then ϕ_N is an M -dilation scaling function and there exists a constant C independent of N such that*

$$\begin{aligned} & \left(\frac{1}{2} - \frac{(M-1) \ln(1 + \frac{1}{M-1})}{2 \ln M} \right) N - \frac{\ln N}{4 \ln M} - C \\ & \leq R(\phi_N) \leq \left(\frac{1}{2} - \frac{(M-1) \ln(1 + \frac{1}{M})}{2 \ln M} \right) N - \frac{\ln N}{4 \ln M} + C. \end{aligned}$$

Remark 1. Observe that $W(\phi_N) \leq 2(M-1)MN$. Therefore the regularity $R(\phi_N)$ of ϕ_N grows proportionally to the filter support width $W(\phi_N)$, i.e., (3) holds.

Remark 2. Let $D(\phi) = R(\phi)/W(\phi)$ be the rate of regularity and filter support width of a scaling function ϕ . Then

$$D(\phi_N) \geq \frac{1}{4M(M-1)} \left(1 - \frac{(M-1) \ln(1 + \frac{1}{M-1})}{\ln M} \right) - C \frac{\ln N}{N},$$

and

$$D_{(N\phi)} \leq \frac{\ln N}{4NM \ln M} + \frac{C}{N}$$

when M is odd, and

$$D_{(N\phi)} \leq \frac{\ln \left(\sin M\pi/(2M+2) \right)^{-1}}{M \ln M} + C \frac{\ln N}{N}$$

when M is even. Therefore we get

$$D(\phi_N)/D_{(N\phi)} \geq \frac{N}{\ln N} \left(\frac{\ln M}{M-1} - \ln \left(1 + \frac{1}{M-1} \right) \right) - C$$

when M is odd, and

$$D(\phi_N)/D_{(N\phi)} \geq \frac{\ln M - (M-1) \ln(1 + \frac{1}{M-1})}{4(M-1) \ln(\sin M\pi/(2M+2))^{-1}} - C \frac{\ln N}{N}$$

when M is even. This shows that $D(\phi_N)$ of the M -dilation scaling function ϕ_N is larger than the one of ${}_N\phi$ even when M is an even integer larger than 4.

2. PROOF OF THE THEOREM

To prove the Theorem, we estimate $H_N(\xi)$ first. Let

$$h(\xi) = \frac{\sin^2 M\xi/2}{M^2 \sin^2 \xi/2}$$

and

$$B_N(\xi) = (h(\xi))^{-N} |H_N(\xi)|^2.$$

Then for all real valued ξ we have

$$\begin{aligned} B_N(-\xi) &= \sum_{k_0+\dots+k_{M-1}=MN-M+1} \alpha_N(k_0, k_{M-1}, \dots, k_1) \frac{(MN - M + 1)!}{k_0!k_1! \dots k_{M-1}!} \\ &\quad \times (h(\xi))^{k_0-N} \prod_{l=1}^{M-1} \left(h(\xi + \frac{2l\pi}{M})\right)^{k_l} \\ &= B_N(\xi) \end{aligned}$$

and

$$B_N(\xi) \geq 0.$$

Therefore by the Riesz Lemma ([D, p.172]) we obtain the existence of $H_N(\xi)$ with

$$H_N(\xi) = \left(\frac{1 - e^{iM\xi}}{M(1 - e^{i\xi})}\right)^N \tilde{H}_N(\xi)$$

and

$$|\tilde{H}_N(\xi)|^2 = B_N(\xi).$$

From the definition of $\alpha_N(k_0, \dots, k_{M-1})$ and from

$$\sum_{l=0}^{M-1} h(\xi + \frac{2l\pi}{M}) = 1,$$

we get

$$\begin{aligned} \alpha_N(k_0, k_1, \dots, k_{M-1}) + \alpha_N(k_1, k_2, \dots, k_{M-1}, k_0) \\ + \dots + \alpha_N(k_{M-1}, k_0, \dots, k_{M-2}) = 1 \end{aligned}$$

and

$$\begin{aligned} &\sum_{l=0}^{M-1} |H_N(\xi + \frac{2l\pi}{M})|^2 \\ &= \sum_{k_0+\dots+k_{M-1}=MN-M+1} (\alpha_N(k_0, k_1, \dots, k_{M-1}) + \alpha_N(k_1, k_2, \dots, k_{M-1}, k_0) \\ &\quad + \dots + \alpha_N(k_{M-1}, k_0, \dots, k_{M-2})) \times \frac{(MN - M + 1)!}{k_0!k_1! \dots k_{M-1}!} \prod_{l=0}^{M-1} \left(h(\xi + \frac{2l\pi}{M})\right)^{k_l} \\ &= \left(\sum_{l=0}^{M-1} h(\xi + \frac{2l\pi}{M})\right)^{MN-M+1} \\ &= 1. \end{aligned}$$

Therefore (4) holds for $H_N(\xi)$. Recall that $H_N(\xi) \neq 0$ when $|\xi| \leq \pi/M$. Hence the solution ϕ_N of (1) corresponding to the symbol $H_N(\xi)$ is an M -dilation scaling function by an elementary argument ([D, p.182, Theorem 6.3.1] with $K = [-\pi, \pi]$).

To estimate the regularity of ϕ_N , we need some estimates on $B_N(\xi)$. From the Stirling formula, which says that $n!$ is equivalent to $n^n e^{-n} \sqrt{n}$, from $1/(M-1) \leq \alpha_N(k_0, \dots, k_{M-1}) \leq 1$ when $k_0 \geq N$ and from $h(2\pi/(M-1)) = 1/M^2$, we get

$$\begin{aligned} B_N(\xi) &\leq \sum_{k_0+\dots+k_{M-1}=MN-M+1, k_0 \geq N} \frac{(MN-M+1)!}{k_0! \cdots k_{M-1}!} \\ &\quad \times (h(\xi))^{k_0-N} \prod_{l=1}^{M-1} (h(\xi + \frac{2l\pi}{M}))^{k_l} \\ &= \sum_{0 \leq k_0 \leq (N-1)(M-1)} \frac{(MN-M+1)!}{(k_0+N)!((N-1)(M-1)-k_0)!} \\ &\quad \times (h(\xi))^{k_0} (1-h(\xi))^{(N-1)(M-1)-k_0} \\ &\leq \frac{(MN-M+1)!}{N!((N-1)(M-1))!} \leq CM^N (1 + \frac{1}{M-1})^{(M-1)N} N^{-1/2} \end{aligned}$$

and

$$\begin{aligned} B_N(2\pi/(M-1)) &\geq \frac{1}{M-1} \sum_{k_1+\dots+k_{M-1}=(M-1)(N-1)} \frac{(MN-M+1)!}{N!k_1! \cdots k_{M-1}!} \\ &\quad \times \prod_{l=1}^{M-1} (h(2\pi/(M-1) + \frac{2l\pi}{M}))^{k_l} \\ &\geq \frac{1}{M-1} \times \frac{(MN-M+1)!}{N!((N-1)(M-1))!} (1 - \frac{1}{M^2})^{(N-1)(M-1)} \\ &\geq CM^N (1 + \frac{1}{M})^{(M-1)N} N^{-1/2}. \end{aligned}$$

Therefore we get

$$R(\phi_N) \geq (\frac{1}{2} - \frac{(M-1) \ln(1 + \frac{1}{M-1})}{2 \ln M})N - \frac{\ln N}{4 \ln M} - C$$

by an argument as in [D, p.217]. Observe that $2M\pi/(M-1) = 2\pi/(M-1) + 2\pi$. By an argument similar to that on p.220 of [D] we obtain

$$R(\phi_N) \leq (\frac{1}{2} - \frac{(M-1) \ln(1 + 1/M)}{2 \ln M})N - \frac{\ln N}{4 \ln M} - C.$$

This completes the proof of the Theorem.

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