PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 12, December 1998, Pages 3501–3506
S 0002-9939(98)05070-9

A CLASS OF M-DILATION SCALING FUNCTIONS WITH REGULARITY GROWING PROPORTIONALLY TO FILTER SUPPORT WIDTH

XIANLIANG SHI AND QIYU SUN

(Communicated by J. Marshall Ash)

Abstract. In this paper, a class of M-dilation scaling functions with regularity growing proportionally to filter support width is constructed. This answers a question proposed by Daubechies on p.338 of her book *Ten Lectures on Wavelets* (1992).

1. Introduction

Let $M \geq 2$ be a fixed integer. A multiresolution analysis for dilation M consists of a sequence of closed subspaces V_j of $L^2(\mathbb{R})$ that satisfy the following conditions (see [C], [D], [M]):

i) $V_j \subset V_{j+1}, \forall j \in \mathbb{Z}$;
ii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$;
iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
iv) $f \in V_j \iff f(2^{-j}\cdot) \in V_0$;
v) there exists a function ϕ in V_0 such that $\{\phi(\cdot - n); n \in \mathbb{Z}\}$ is an orthonormal basis of V_0.

The function ϕ is called an M-dilation scaling function. It is easy to see that ϕ satisfies the refinement equation

$$\phi(x) = \sum_{n\in\mathbb{Z}} c_n \phi(Mx - n), \tag{1}$$

where the sequence $\{c_n\}$ satisfies

$$\sum_{n\in\mathbb{Z}} c_n = M.$$

Received by the editors November 20, 1995.
1991 *Mathematics Subject Classification.* Primary 42C15.
The first author is supported by the Texas Higher Education Coordinating Board under Grant Number 999903-109. The second author is partially supported by the National Natural Sciences Foundation of China # 69735020, the Tian Yuan Foundation, the Doctoral Bases Promotion Foundation of National Educational Commission of China # 97033519 and the Zhejiang Provincial Sciences Foundation of China # 196083, and by the Wavelets Strategic Research Program, National University of Singapore, under a grant from the National Science and Technology Board and the Ministry of Education, Singapore.

In this paper we shall only deal with compactly supported M-dilation scaling functions. In this case the sequence $\{c_n\}$ must have finite length. The function

$$H(\xi) = \frac{1}{M} \sum_{n \in \mathbb{Z}} c_n e^{in\xi} \tag{2}$$

is called a symbol corresponding to the refinement equation (1).

The filter support width $W(\phi)$ of an M-dilation scaling function ϕ is defined as the difference of the largest and the smallest indices of the nonzero c_n. The regularity $R(\phi)$ of ϕ is defined as the supremum of α such that $\phi \in C^\alpha$, where C^α denotes the Hölder class of index α.

In her book [D, p.338], Daubechies remarks that:

> *At present, I know of no explicit scheme that provides an infinite family of* m_0 (i.e., symbol H), *for dilation* 3 (i.e., $M = 3$), *with regularity growing proportionally to the filter support width.*

To our knowledge, this question is still open. The purpose of this paper is to construct a class of M-dilation scaling functions ϕ_N for which there exists a constant λ_M independent of N such that

$$R(\phi_N) \geq \lambda_M W(\phi_N), \tag{3}$$

where $M \geq 3$. On the other hand it is already known that (see [DL])

$$W(\phi_N) \geq R(\phi_N).$$

These facts give an affirmative answer to Daubechies' question.

The regularity of ϕ has been studied widely; see for instance [CL], [BDS], [D], [HW1], [HW2], [So], [S] and [WL]. In general, to study the regularity of ϕ we need to consider the symbol (2) first. By the Fourier transform, we see that all the symbols H satisfy

$$\sum_{l=0}^{M-1} |H(\xi + \frac{2l\pi}{M})|^2 = 1. \tag{4}$$

The solutions H of the equation (4) are determined by (see [BDS], [H], [HSZ])

$$|H(\xi)|^2 = \Big(\frac{\sin^2(M\xi/2)}{M^2 \sin^2 \xi/2}\Big)^N \sum_{s=0}^{N-1} {}^M_N a(s) \sin^{2s} \frac{\xi}{2} + (\sin \frac{M\xi}{2})^{2N} R(\xi), \tag{5}$$

where

$${}^M_N a(s) = \sum_{s_1+\cdots+s_{M-1}=s} \prod_{j=1}^{M-1} \begin{pmatrix} N-1+s_j \\ s_j \end{pmatrix} \frac{1}{\sin^{2s_j} j\pi/M}$$

and R is a real-valued trigonometric polynomial such that $\sum_{l=0}^{M-1} R(\xi + 2l\pi/M) = 0$ and the right hand side of (5) is nonnegative.

By the Riesz Lemma (see [D, p.172]), such symbol H exists. Let ${}_NH$ be a solution of (5) with $R = 0$, and let ${}_N\phi$ be the solution of (1) corresponding to the symbol ${}_NH$. In [BDS], Bi, Dai and Sun prove the following estimates on the regularity of ${}_N\phi$:

$$|R({}_N\phi) - \frac{\ln N}{4 \ln M}| \leq C$$

when M is odd, and

$$|R({}_N\phi) - \frac{4N\ln\left(\sin\frac{M\pi}{2M+2}\right)^{-1} + \ln N}{4\ln M}| \le C$$

when M is even. A more precise estimate of $R({}_N\phi)$ can be found in [S]. For the special cases $M = 3, 4, 5$, similar results are obtained by Soardi ([So]) and Heller and Wells ([HW2]) independently. This result shows that for these special ${}_N\phi$ the regularity does not grow proportionally to the filter support width when M is odd. To construct M-dilation scaling functions with regularity growing proportionally to the filter support width we use the symbol H_N determined by

$$\begin{aligned} |H_N(\xi)|^2 &= \sum_{k_0+\cdots+k_{M-1}=MN-M+1} \alpha_N(k_0,\cdots,k_{M-1})\frac{(MN-M+1)!}{k_0!\cdots k_{M-1}!} \\ &\quad \times \prod_{l=0}^{M-1}\left(\frac{\sin M\xi/2}{M\sin(\xi/2+l\pi/M)}\right)^{2k_l}, \end{aligned}$$

where $N \ge 1$, and $\alpha_N(k_0,\cdots,k_{M-1})$ is defined by

$$\alpha_N(k_0,\cdots,k_{M-1}) = \begin{cases} 0, & \text{if} \quad k_0 \le N-1, \\ \frac{1}{\#(E)}, & \text{if} \quad k_0 \ge N, \end{cases}$$

where $E = \{j : k_j \ge N\}$ and $\#(E)$ is the cardinality of E. Let ϕ_N be the solution of (1) corresponding to a symbol H_N. Then we have the following

Theorem. *Let $M \ge 3$ and $N \ge 2$ be any natural numbers. Then ϕ_N is an M-dilation scaling function and there exists a constant C independent of N such that*

$$\begin{aligned} &\left(\frac{1}{2} - \frac{(M-1)\ln(1+\frac{1}{M-1})}{2\ln M}\right)N - \frac{\ln N}{4\ln M} - C \\ \le\ & R(\phi_N) \le \left(\frac{1}{2} - \frac{(M-1)\ln(1+\frac{1}{M})}{2\ln M}\right)N - \frac{\ln N}{4\ln M} + C. \end{aligned}$$

Remark 1. Observe that $W(\phi_N) \le 2(M-1)MN$. Therefore the regularity $R(\phi_N)$ of ϕ_N grows proportionally to the filter support width $W(\phi_N)$, i.e., (3) holds.

Remark 2. Let $D(\phi) = R(\phi)/W(\phi)$ be the rate of regularity and filter support width of a scaling function ϕ. Then

$$D(\phi_N) \ge \frac{1}{4M(M-1)}\left(1 - \frac{(M-1)\ln(1+\frac{1}{M-1})}{\ln M}\right) - C\frac{\ln N}{N},$$

and

$$D({}_N\phi) \le \frac{\ln N}{4NM\ln M} + \frac{C}{N}$$

when M is odd, and

$$D({}_N\phi) \le \frac{\ln\left(\sin M\pi/(2M+2)\right)^{-1}}{M\ln M} + C\frac{\ln N}{N}$$

when M is even. Therefore we get

$$D(\phi_N)/D({}_N\phi) \ge \frac{N}{\ln N}\left(\frac{\ln M}{M-1} - \ln\left(1+\frac{1}{M-1}\right)\right) - C$$

when M is odd, and

$$D(\phi_N)/D({}_N\phi) \ge \frac{\ln M - (M-1)\ln(1+\frac{1}{M-1})}{4(M-1)\ln(\sin M\pi/(2M+2))^{-1}} - C\frac{\ln N}{N}$$

when M is even. This shows that $D(\phi_N)$ of the M-dilation scaling function ϕ_N is larger than the one of ${}_N\phi$ even when M is an even integer larger than 4.

2. Proof of the Theorem

To prove the Theorem, we estimate $H_N(\xi)$ first. Let

$$h(\xi) = \frac{\sin^2 M\xi/2}{M^2 \sin^2 \xi/2}$$

and

$$B_N(\xi) = \big(h(\xi)\big)^{-N} |H_N(\xi)|^2.$$

Then for all real valued ξ we have

$$\begin{aligned} B_N(-\xi) &= \sum_{k_0+\cdots+k_{M-1}=MN-M+1} \alpha_N(k_0, k_{M-1}, \cdots, k_1) \frac{(MN-M+1)!}{k_0!k_1!\cdots k_{M-1}!} \\ &\qquad \times (h(\xi))^{k_0-N} \prod_{l=1}^{M-1} (h(\xi + \frac{2l\pi}{M}))^{k_l} \\ &= B_N(\xi) \end{aligned}$$

and

$$B_N(\xi) \ge 0.$$

Therefore by the Riesz Lemma ([D, p.172]) we obtain the existence of $H_N(\xi)$ with

$$H_N(\xi) = \big(\frac{1-e^{iM\xi}}{M(1-e^{i\xi})}\big)^N \tilde{H}_N(\xi)$$

and

$$|\tilde{H}_N(\xi)|^2 = B_N(\xi).$$

From the definition of $\alpha_N(k_0, \cdots, k_{M-1})$ and from

$$\sum_{l=0}^{M-1} h(\xi + \frac{2l\pi}{M}) = 1,$$

we get

$$\begin{aligned} &\alpha_N(k_0, k_1, \cdots, k_{M-1}) + \alpha_N(k_1, k_2, \cdots, k_{M-1}, k_0) \\ &\qquad\qquad + \cdots + \alpha_N(k_{M-1}, k_0, \cdots, k_{M-2}) = 1 \end{aligned}$$

and

$$\begin{aligned} &\sum_{l=0}^{M-1} |H_N(\xi + \frac{2l\pi}{M})|^2 \\ = &\sum_{k_0+\cdots+k_{M-1}=MN-M+1} \big(\alpha_N(k_0, k_1, \cdots, k_{M-1}) + \alpha_N(k_1, k_2, \cdots, k_{M-1}, k_0) \\ &\quad + \cdots + \alpha_N(k_{M-1}, k_0, \cdots, k_{M-2})\big) \times \frac{(MN-M+1)!}{k_0!k_1!\cdots k_{M-1}!} \prod_{l=0}^{M-1} \big(h(\xi + \frac{2l\pi}{M})\big)^{k_l} \\ = &\big(\sum_{l=0}^{M-1} h(\xi + \frac{2l\pi}{M})\big)^{MN-M+1} \\ = &\ 1. \end{aligned}$$

Therefore (4) holds for $H_N(\xi)$. Recall that $H_N(\xi) \neq 0$ when $|\xi| \leq \pi/M$. Hence the solution ϕ_N of (1) corresponding to the symbol $H_N(\xi)$ is an M-dilation scaling function by an elementary argument ([D, p.182, Theorem 6.3.1] with $K = [-\pi, \pi]$).

To estimate the regularity of ϕ_N, we need some estimates on $B_N(\xi)$. From the Stirling formula, which says that $n!$ is equivalent to $n^n e^{-n}\sqrt{n}$, from $1/(M-1) \leq \alpha_N(k_0, \cdots, k_{M-1}) \leq 1$ when $k_0 \geq N$ and from $h(2\pi/(M-1)) = 1/M^2$, we get

$$\begin{aligned} B_N(\xi) &\leq \sum_{k_0+\cdots+k_{M-1}=MN-M+1, k_0\geq N} \frac{(MN-M+1)!}{k_0!\cdots k_{M-1}!} \\ &\qquad \times (h(\xi))^{k_0-N} \prod_{l=1}^{M-1} (h(\xi + \frac{2l\pi}{M}))^{k_l} \\ &= \sum_{0\leq k_0\leq (N-1)(M-1)} \frac{(MN-M+1)!}{(k_0+N)!((N-1)(M-1)-k_0)!} \\ &\qquad \times (h(\xi))^{k_0}(1-h(\xi))^{(N-1)(M-1)-k_0} \\ &\leq \frac{(MN-M+1)!}{N!((N-1)(M-1))!} \leq CM^N(1+\frac{1}{M-1})^{(M-1)N}N^{-1/2} \end{aligned}$$

and

$$\begin{aligned} B_N(2\pi/(M-1)) &\geq \frac{1}{M-1} \sum_{k_1+\cdots+k_{M-1}=(M-1)(N-1)} \frac{(MN-M+1)!}{N!k_1!\cdots k_{M-1}!} \\ &\qquad \times \prod_{l=1}^{M-1} (h(2\pi/(M-1) + \frac{2l\pi}{M}))^{k_l} \\ &\geq \frac{1}{M-1} \times \frac{(MN-M+1)!}{N!((N-1)(M-1))!}(1-\frac{1}{M^2})^{(N-1)(M-1)} \\ &\geq CM^N(1+\frac{1}{M})^{(M-1)N}N^{-1/2}. \end{aligned}$$

Therefore we get

$$R(\phi_N) \geq (\frac{1}{2} - \frac{(M-1)\ln(1+\frac{1}{M-1})}{2\ln M})N - \frac{\ln N}{4\ln M} - C$$

by an argument as in [D, p.217]. Observe that $2M\pi/(M-1) = 2\pi/(M-1) + 2\pi$. By an argument similar to that on p.220 of [D] we obtain

$$R(\phi_N) \leq (\frac{1}{2} - \frac{(M-1)\ln(1+1/M)}{2\ln M})N - \frac{\ln N}{4\ln M} - C.$$

This completes the proof of the Theorem.

References

[C] C. K. Chui, *An Introduction to Wavelets*, Academic Press, 1992. MR **93f:**42055

[CL] C. K. Chui and J. Lian, Construction of compactly supported symmetric and antisymmetric orthonormal wavelets with scale = 3, *Appl. Comput. Harmonic Anal.*, 2(1995), 21-51. MR **95m:**42042

[BDS] N. Bi, X. Dai and Q. Sun, Construction of compactly supported M-band wavelets, *Appl. Comput. Harmonic Anal.*, To appear.

[D] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics 61, Philadephia, 1992. MR **93e:**42045

[DL] I. Daubechies and J. Lagarias, Two-scale difference equation I: existence and global regularity of solution, *SIAM J. Math. Anal.*, 22(1991), 1388-1410. MR **92d:**39001

[H] P. N. Heller, Rank M wavelets with N vanishing moments, *SIAM J. Matrix Anal. Appl.*, 16(1995), 502-519. MR **95k:**42058

[HW1] P. N. Heller and R. O. Wells, Jr., The spectral theory of multiresolution operators and applications, In *Wavelets: Theory, Algorithms, and Applications*, edited by C. K. Chui, L. Montefusco, and L. Puccio, Academic Press, 1994, pp. 13-42. MR **96a:**42046

[HW2] P. N. Heller ad R. O. Wells Jr., Sobolev regularity for rank M wavelets, CML Technical Report TR 96-08, Computational Mathematics Laboratory, Rice University, 1996.

[HSZ] D. Huang, Q. Sun and Z. Zhang, Integral representation of M-band filter of Daubechies type, *Chinese Sciences Bulletin*, 42(1997), 803–807. CMP 97:17

[M] Y. Meyer, *Ondelettes et Opérateurs,I: Ondelettes*, Hermann, Paris, 1990. MR **93i:**42002

[So] P. M. Soardi, Hölder regularity of compactly supported p-wavelets, *Constr. Approx.*, To appear.

[S] Q. Sun, Sobolev exponent estimates and asymptotic regularity of M band Daubechies' scaling functions, Preprint, 1997.

[WL] G. V. Welland and M. Lundberg, Construction of compact p-wavelets, *Constr. Approx.*, 9(1993), 347-370. MR **94i:**42049

Department of Mathematics, Texas A&M University, College Station, Texas 77843-3368

E-mail address: `xshi@math.tamu.edu`

Center for Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310027, People's Republic of China

Current address: Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260, Singapore

E-mail address: `matsunqy@leonis.nus.edu.sg`