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EUCLIDEAN CONFORMALLY FLAT SUBMANIFOLDS IN CODIMENSION TWO OBTAINED AS INTERSECTIONS

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ABSTRACT. We characterize a large family of codimension two euclidean conformally flat submanifolds in the class of isometrically rigid ones and construct explicit examples.

In [DF] we showed that generic conformally flat submanifolds with codimension two in euclidean space, $f\colon M^n \to \mathbb{R}^{n+2}$, $n \geq 5$, can be divided into three classes, namely, the surface-like ones, those which admit locally a continuous 1-parameter family of isometric deformations, and those which are locally isometrically rigid. In addition, we generated explicit examples of elements in the first and second classes. On the other hand, no example came out from our rather elaborate descriptions of submanifolds in the third class, thus naturally raising the question of whether such submanifolds really exist. Our main achievement here is give a positive answer to this question. In what follows we make free use of definitions and results in [DF].

Our approach is the following. First we give a parametric description of a large family of generic conformally flat submanifolds, each of which is completely determined by two curves and two functions in one variable. Then, we prove that almost all elements in the family are locally isometrically rigid. Next, we give a brief indication of how they can be obtained by a geometric procedure, namely, as intersections starting from two flat hypersurfaces. We conclude this note by showing that a particular selection of curves and functions yields explicit examples.

Consider a pair of smooth curves $\alpha_j\colon I_j \subset \mathbb{R} \to \mathbb{R}^{n+2}$, $1 \leq j \leq 2$, with $\|\alpha_1(u)\| > 1$ everywhere. Now take $V^2 = I_1 \times I_2$ small enough so that

$$\theta(u,v) = \theta(u,0) + \int_0^v \theta_v(u,s)ds, \qquad \theta(u,0) = \sqrt{\|\alpha_1(u)\|^2 - 1}, \tag{1}$$

is positive, θ_v being the unique (and necessarily smooth) solution of the integral equation of Volterra type

$$\begin{aligned} \theta_v(u,v) = &- \langle \alpha_1(u), \alpha_2(v)\rangle - \theta(u,0)\int_0^v \langle \alpha_2(v), \alpha_2(t)\rangle dt \\ &- \int_0^v \left(\int_s^v \langle \alpha_1(v), \alpha_2(t)\rangle dt\right)\theta_v(u,s)ds. \end{aligned} \tag{2}$$

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We claim that $h\colon V^2 \to \mathbb{R}^{n+2}$, parametrized by

$$h(u,v) = \frac{1}{\sqrt{1+\theta^2}}\left(\alpha_1(u) + \int_0^v \theta(u,s)\alpha_2(s)\,ds\right), \tag{3}$$

satisfies $h(V) \subset \mathbb{S}_1^{n+1} \subset \mathbb{R}^{n+2}$. In fact, this is equivalent to

$$1 + \theta^2(u,v) = \|\alpha_1(u) + \int_0^v \theta(u,s)\alpha_2(s)\,ds\|^2.$$

Taking derivatives with respect to v yields

$$\theta_v(u,v) = \langle \alpha_1(u), \alpha_2(v)\rangle + \int_0^v \langle \alpha_2(v), \alpha_2(s)\rangle \theta(u,s)\,ds. \tag{4}$$

Now it is not difficult to see, using (1) and Fubini's theorem, that (2) and (4) are equivalent. Thus the claim follows easily.

Define $\beta\colon V^2 \to \mathbb{R}^{n+2}$ as

(5)

$$\beta = \rho\theta h - \int_0^v\int_0^u \theta(s,t)b(s)\alpha_2(t)\,ds\,dt - \int_0^v a(t)\alpha_2(t)\,dt - \int_0^u b(s)\alpha_1(s)\,ds,$$

where $\rho \in C^\infty(V)$ is given by

$$\rho(u,v) = \frac{1}{\sqrt{1+\theta^2}}\left(a(v) + \int_0^u \theta(s,v)b(s)\,ds\right) \tag{6}$$

and $a(u), b(v)$ are arbitrary smooth functions. Finally, consider V^2 endowed with the metric induced by h, and let Θ^* stand for the adjoint to the tensor $\Theta\colon TV \to TV$ defined as

$$\Theta\partial_u = \frac{1}{\theta}\partial_u, \;\; \Theta\partial_v = -\theta\partial_v.$$

Theorem. *The map $\varphi\colon \mathcal{N}_1 \to \mathbb{R}^{n+2}$, defined on the unit normal bundle $\mathcal{N}_1$ in $\mathbb{S}_1^{n+1}$ of h as in (3), and given by*

$$\varphi(\eta) = \beta - \Theta^* \operatorname{grad}\rho + \sqrt{\rho^2 - \|\Theta^* \operatorname{grad}\rho\|^2}\,\eta,$$

is, along the open subset of regular points, a parametrization of a generic $n-$dimensional conformally flat submanifold.

Assume, in addition, that $\alpha_1(u)$ is not locally contained in any subspace $\mathbb{R}^4 \subset \mathbb{R}^{n+2}$ and that on no open subset of V^2 do the curves $\alpha_1(u)$, $\alpha_2(v)$ satisfy

$$\alpha_1(u) = \alpha_0(u) + \sqrt{|1 - \|\alpha_0(u)\|^2|}\,\xi_0, \;\; \xi_0 \in \mathbb{R}^{n+1}, \tag{7}$$

with $\langle \alpha_0(u), \xi_0\rangle = \langle \alpha_0(u), \alpha_2(v)\rangle = 0$. Then the submanifold parametrized by φ is locally isometrically rigid.

Proof. First, notice that $\|\alpha_1(u)\| > 1$ implies that $\|\alpha_0(u)\| \neq 1$. Now set $\tau(u,v) = -\theta^2(u,v)$. A straightforward computation yields

$$h_{vu} = \frac{\tau_v}{2(1-\tau)}h_u + \frac{\tau_u}{2\tau(1-\tau)}h_v + \frac{2\tau\tau_{uv} + (2\tau-1)\tau_u\tau_v}{4\tau(1-\tau)^2}h. \tag{8}$$

Hence, the coordinates (u,v) are conjugate and τ is a negative solution of

$$\begin{cases} \tau_u = 2\Gamma^2\tau(1-\tau), \\ \tau_v = 2\Gamma^1(1-\tau), \end{cases} \tag{9}$$

where Γ^1, Γ^2 are the Christoffel symbols defined by $\nabla'_{h_u} h_v = \Gamma^1 h_u + \Gamma^2 h_v$. Moreover, a straightforward computation shows that $\beta(u,v)$ in (5) is a solution of the system

$$\begin{cases} \beta_u = \theta\rho h_u - \dfrac{\rho_u}{\theta} h, \\ \beta_v = -\dfrac{\rho}{\theta} h_v + \theta\rho_v h, \end{cases}$$

and our first statement follows from Theorem 10 in [DF].

We have from our assumption on α_1 that the image $h(V_0)$ of any open subset $V_0 \subset V$ can never be contained in a totally geodesic sphere $\mathbb{S}^3_1 \subset \mathbb{S}^{n+1}_1$. Thus, by [DF] the submanifold is nowhere surface-like. From (8),

$$F = \langle \partial_u, \partial_v \rangle = -\frac{2\tau\tau_{uv} + (2\tau - 1)\tau_u\tau_v}{4\tau(1-\tau)^2},$$

which is easily seen to imply that h satisfies the additional condition

$$\Gamma^1_u - \Gamma^1\Gamma^2 + F = 0. \tag{10}$$

Then (9) and (10) yield

$$\left(\frac{\tau_u}{\tau}\right)_v = 2(1-\tau)(\Gamma^2_v - 2\Gamma^1\Gamma^2).$$

By definition, h is of first type if and only if $\Gamma^2_v - 2\Gamma^1\Gamma^2 = 0$. Hence, being of first type is equivalent to the existence of functions $U(u)$ and $V(v)$ satisfying

$$\theta(u,v) = \frac{V(v)}{U(u)}. \tag{11}$$

Substituting (11) in (3) gives

$$U^2(u) + V^2(v) = \|U(u)\,\alpha_1(u) + \int_0^v V(s)\,\alpha_2(s)\,ds\|^2. \tag{12}$$

It follows easily that

$$\langle (U(u)\,\alpha_1(u))', \alpha_2(v)\rangle = 0.$$

Hence, we have that

$$\alpha_1(u) = \alpha_0(u) + \frac{1}{U(u)}\,\xi_0, \quad \xi_0 \in \mathbb{R}^{n+1}, \tag{13}$$

where $\langle \alpha_0(u), \xi_0\rangle = \langle \alpha_0(u), \alpha_2\rangle = 0$. From (12) and (13),

$$U^2(u)(1 - \|\alpha_0(u)\|^2) + V^2(v) - \|\xi_0 + \int_0^v V(s)\,\alpha_2(s)\,ds\|^2 = 0.$$

Thus, there exists a constant $k \neq 0$ such that

$$U^2(u) = \frac{k}{1 - \|\alpha_0(u)\|^2}, \quad V^2(v) = -k + \|\xi_0 + \int_0^v V(s)\,\beta(s)\,ds\|^2. \tag{14}$$

In view of (11), we can assume without loss of generality that $k = \pm 1$. To conclude the proof we have to show that the second equation in (14) always has a solution. By a procedure similar to the one used above, this is equivalent to proving that a certain integral equation of Volterra type in $V'(v)$ can be solved. The rest of the proof follows from Theorem 7 in [DF]. $\square$

Remark. One can easily conclude from the Theorem that there exist connected generic conformally flat submanifolds M^n in $\mathbb{R}^{n+2}$ which are locally rigid on one open subset and admit a 1-parameter family of local isometric deformations on another open subset.

By arguments similar to those used in [DFT], one can show that the conformally flat submanifolds given in the first part of the Theorem are precisely the ones which can be obtained by the following geometric procedure.

Consider a pair of embedded flat hypersurfaces in flat Lorentzian space $F\colon U_1 \subset \mathbb{R}^{n+2} \to \mathbf{L}^{n+3}$ and $G\colon U_2 \subset \mathbf{L}^{n+2} \to \mathbf{L}^{n+3}$ of rank one, i.e., free of totally geodesic points, so that their $(n+1)$-dimensional relative nullity spaces Δ_F, Δ_G are transversal at any point of $N^{n+1} = F(U_1) \cap G(U_2)$. Now, set

$$F^0 = (F|_{F^{-1}(N^{n+1})})^{-1}\colon\ N^{n+1} \to U_1 \subset \mathbb{R}^{n+2}$$

and

$$G^0 = (G|_{G^{-1}(N^{n+1})})^{-1}\colon\ N^{n+1} \to U_2 \subset \mathbf{L}^{n+2}.$$

Then, the intersection

$$M^n = G^0(N^{n+1}) \cap \mathbf{V}^{n+1}$$

of $G^0(N^{n+1})$ with the light cone $\mathbf{V}^{n+1} \subset \mathbf{L}^{n+2}$ yields, generically, a conformally flat submanifold $f\colon\ M^n \to U_1 \subset \mathbb{R}^{n+2}$ given by the construction in the Theorem as

$$f := (F^0|_{(F^0)^{-1}(M^n)})^{-1}\colon\ M^n \to \mathbb{R}^{n+2}.$$

We conclude this note showing that our Theorem can be used to construct explicit examples.

Examples. Let $\gamma\colon\ I_1 \to \mathbb{R}^{n+1}$ be a smooth curve parametrized by arclength, not locally contained in $\mathbb{R}^3$, which satisfies $\|\gamma(u)\|^2 = 2$. Define $\alpha_1\colon I_1 \to \mathbb{R}^{n+2}$ by

$$\alpha_1(u) = \gamma(u) + \sinh(u)\, e, \quad e = e_{n+2}.$$

Take $\alpha_2\colon\ I_2 \to \mathbb{R}^{n+2}$ as

$$\alpha_2(v) = e.$$

An easy computation shows that

$$\theta(u,v) = \cosh(u+v).$$

Choose $a(v) = 1$, $b(u) = 0$ in (6). Hence,

$$h(u,v) = \rho(u,v)(\gamma(u) + \sinh(u+v)\, e)$$

and

$$\beta(u,v) = \rho\theta h(u,v) - ve,$$

where $\rho(u,v) = (1+\theta^2(u,v))^{-1/2}$. A straightforward computation yields

$$\Theta^* \operatorname{grad} \rho = \frac{1}{2}\, \rho^2\theta(1-\theta^2)\gamma - \theta_u\gamma' + \rho^2\theta\theta_u\, e.$$

Thus,

$$r = \rho^2 - \|\Theta^* \operatorname{grad}\, \rho\|^2 = \frac{1}{2}(4-3\theta^2) = \frac{1}{2}(1-3\sinh^2(u+v))$$

is positive for small $V^2 = I_1 \times I_2$. The unit normal bundle of h in $\mathbb{S}_1^{n+1}$ is

$$\mathcal{N}_1(u,v) = \{\eta \in \mathbb{S}_1^{n+1} \subset \mathbb{R}^{n+2}\colon\ \eta \perp \operatorname{span}\{\gamma(u), \gamma'(u), e\}\}.$$

Then, on the open subset $\mathcal{U} \subset \mathcal{N}_1$ of regular points, the map $\psi\colon \mathcal{N}_1 \to \mathbb{R}^{n+2}$ given by

$$\psi(\eta) = \frac{1}{2}\cosh(u+v)\,\gamma + \sinh(u+v)\,\gamma' - v\,e + \sqrt{r}\,\eta$$

is a parametrization of a generic n-dimensional conformally flat submanifold in $\mathbb{R}^{n+2}$ which, by the second part of the Theorem, is locally isometrically rigid.

It remains to show that $\mathcal{U}$ is not empty. We have that

$$\psi_u = \frac{1}{2\sqrt{r}}(2\sqrt{r}\,\langle \eta, \gamma''\rangle - 3\theta)(\theta_u \eta - \sqrt{r}\,\gamma') + \xi,$$

where $\xi \in T^{\perp}_{h(u,v)}V$, $\xi \perp \eta$. On the other hand,

$$\psi_v = \delta - e, \;\; \delta \perp e.$$

We easily conclude that

$$\mathcal{U} = \{\eta \in \mathcal{N}_1 \colon\; 2\sqrt{r}\,\langle \eta, \gamma''\rangle - 3\theta \neq 0\},$$

as we wished.

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