PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 1, January 1999, Pages 251–257
S 0002-9939(99)04512-8

LUSIN SETS

MARION SCHEEPERS

(Communicated by Andreas R. Blass)

ABSTRACT. We show that a set of real numbers is a Lusin set if, and only if, it has a covering property similar to the familiar property of Rothberger

In [7] Rothberger defined the property C'': A set X of real numbers has property C'' if: For every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(U_n : n \in \mathbb{N})$ such that for each n $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\}$ is a cover for X. This is an example of the following selection hypothesis: Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of an infinite set. Then $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes: For every sequence $(O_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(T_n : n \in \mathbb{N})$ such that $T_n \in O_n$ for each n, and $\{T_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$. The game $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ associated with this hypothesis is played as follows: ONE and TWO play an inning per positive integer. In the n–th inning ONE first chooses a set $O_n \in \mathcal{A}$, and TWO responds with a $T_n \in O_n$. TWO wins a play $O_1, T_1, \dots, O_n, T_n, \dots$ if $\{T_n : n \in \mathbb{N}\}$ is in $\mathcal{B}$; otherwise, ONE wins.

Let (X, τ) be a topological space which is at least T_3. Define:

$\mathcal{K}$ is the set of those $\mathcal{U} \subset \tau$ such that $X = \bigcup\{\overline{U} : U \in \mathcal{U}\}$.

$\mathcal{K}_\Omega$ is the set of $\mathcal{U}$ in $\mathcal{K}$ such that no element of $\mathcal{U}$ is dense in X, and for each finite set $F \subseteq X$, there is a $U \in \mathcal{U}$ such that $F \subseteq \overline{U}$.

$\mathcal{O}$ is the collection of all open covers of X.

In this notation property C'' is $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$. A T_3–space has property $\mathsf{S}_1(\mathcal{O}, \mathcal{K})$ if, and only if, it has property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$. If a T_3–space has property $\mathsf{S}_1(\mathcal{K}, \mathcal{K})$, then it has property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ (since $\mathcal{O} \subset \mathcal{K}$).

A set of real numbers is a *Lusin set* if it is uncountable but its intersection with every first category set is countable. In [5] Lusin showed that the Continuum Hypothesis implies the existence of a Lusin set. In Theorem 3.18 of [4] Kunen shows that for each regular uncountable cardinal number κ it is consistent that the real line has cardinality κ, and there is a Lusin set of cardinality κ. Lusin sets have property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$. We shall show that a set of real numbers is a Lusin set if, and only if, it has property $\mathsf{S}_1(\mathcal{K}, \mathcal{K})$ (Theorem 2). It will follow that not all subsets of $\mathbb{R}$ having property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ have property $\mathsf{S}_1(\mathcal{K}, \mathcal{K})$.

A related selection hypothesis, denoted by $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$, is defined as follows: For every sequence $(O_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is a sequence $(T_n : n \in \mathbb{N})$ such that for each n T_n is a finite subset of O_n and $\bigcup_{n=1}^{\infty} T_n$ is in $\mathcal{B}$. The game $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$

Received by the editors November 5, 1996 and, in revised form, May 16, 1997.
1991 *Mathematics Subject Classification.* Primary 90D44.
Key words and phrases. Lusin set, infinite game, partition relation.
The author's research was funded in part by NSF grant DMS 95-05375.

associated with this hypothesis is played as follows: ONE and TWO play an inning per positive integer. In the n–th inning ONE first chooses an $O_n \in \mathcal{A}$, after which TWO chooses a finite set $T_n \subseteq O_n$. TWO wins a play $O_1, T_1, \dots, O_n, T_n, \dots$ if $\bigcup_{n=1}^{\infty} T_n \in \mathcal{B}$; otherwise, ONE wins. It is well–known that the property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ is stronger than the property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$. We shall see that for subsets of the real line $\mathsf{S}_1(\mathcal{K},\mathcal{K})$ and $\mathsf{S}_{fin}(\mathcal{K},\mathcal{K})$ are equivalent.

We shall need a connection with the following notion, due to Recław: A subset X of $\mathbb{R}$ is said to be an $R^{\mathcal{M}}$–set if for every Borel subset B of $\mathbb{R} \times {}^{\mathbb{N}}\mathbb{N}$ such that for each $x \in X$ the set $B_x := \{f : (x,f) \in B\}$ is of the first category, $\bigcup_{x\in X} B_x$ is not all of ${}^{\mathbb{N}}\mathbb{N}$. Since a Borel subset of a subspace is always the intersection of the subspace with a Borel subset of the superspace, we may in this definition restrict attention to Borel subsets of $X \times {}^{\mathbb{N}}\mathbb{N}$ instead of $\mathbb{R} \times {}^{\mathbb{N}}\mathbb{N}$.

Theorem 1 (Recław, [6])**.** *Lusin sets are $R^{\mathcal{M}}$–sets.*

A characterization of Lusin sets

A space is $\mathcal{K}$*–Lindelöf* if each element of $\mathcal{K}$ has a countable subset in $\mathcal{K}$.

Theorem 2. *If $X \subseteq \mathbb{R}$ is uncountable, then the following are equivalent:*

1. *X has property $\mathsf{S}_1(\mathcal{K},\mathcal{K})$.*
2. *X has property $\mathsf{S}_{fin}(\mathcal{K},\mathcal{K})$.*
3. *X is $\mathcal{K}$–Lindelöf.*
4. *X is a Lusin set.*
5. *ONE has no winning strategy in the game $\mathsf{G}_1(\mathcal{K},\mathcal{K})$.*
6. *ONE has no winning strategy in the game $\mathsf{G}_{fin}(\mathcal{K},\mathcal{K})$.*

Proof. The proofs of $1 \Rightarrow 2$, $2 \Rightarrow 3$, $5 \Rightarrow 6$, $5 \Rightarrow 1$ and $6 \Rightarrow 2$ are standard. We show that $3 \Leftrightarrow 4$ and $3 \Rightarrow 5$.

$\underline{3 \Rightarrow 4}$: Assume that X is uncountable, but not a Lusin set. Let C be an uncountable closed, nowhere dense set such that $X \cap C$ is uncountable. Define an element of $\mathcal{K}$ for X as follows: For $y \in X \setminus C$, let V_y be a neighborhood of y with $\overline{V_y}$ disjoint from C. For $y \in X \cap C$, choose for each n an open interval $I_n(y)$ in $(y - \frac{1}{2^n}, y + \frac{1}{2^n})$ such that $\overline{I_n(y)} \cap C = \emptyset$. Then put $V_y = \bigcup_{n<\infty} I_n(y)$. The set $\mathcal{U} = \{V_y : y \in X\}$ is in $\mathcal{K}$. Also, for each y such that $\overline{V_y} \cap C \neq \emptyset$ we have $\overline{V_y} \cap C = \{y\}$. Since $X \cap C$ is uncountable, no countable subset of $\mathcal{U}$ is in $\mathcal{K}$ for X. Thus, X is not $\mathcal{K}$–Lindelöf.

$\underline{4 \Rightarrow 3}$: Let $X \subseteq \mathbb{R}$ be a Lusin set and let $\mathcal{U}$ be an element of $\mathcal{K}$ for X. We may assume that X is dense in $\mathbb{R}$. Let $D \subset X$ be a countable dense subset of X which is contained in $\bigcup \mathcal{U}$. For each $d \in D$ pick an element U_d of $\mathcal{U}$ which contains it. Since $X \setminus \bigcup_{d\in D} U_d$ is nowhere dense, it is countable. For each element d of this countable set, choose an element $U_d \in \mathcal{U}$ with $d \in \overline{U}_d$. Then the collection of all U_d's we selected during these two stages is a countable set in $\mathcal{K}$ for X.

$\underline{3 \Rightarrow 5}$: Let F be a strategy for ONE. By 3 X is $\mathcal{K}$–Lindelöf; thus we may assume that in each inning F requires that ONE plays a countable element of $\mathcal{K}$. Using F construct the following array $(U_\sigma : \sigma \in {}^{<\omega}\mathbb{N})$ of open subsets of X: $(U_n : n \in \mathbb{N})$ enumerates ONE's first move, $F(\emptyset)$; $(U_{n_1,n} : n \in \mathbb{N})$ enumerates $F(U_{n_1})$, $(U_{n_1,n_2,n} : n \in \mathbb{N})$ enumerates $F(U_{n_1}, U_{n_1,n_2})$, and so on. The array has the property that for each σ, $X = \bigcup_{n\in\mathbb{N}} \overline{U}_{\sigma\frown n}$.

First, define a subset C of $X \times {}^{\mathbb{N}}\mathbb{N}$ by: $C = \{(x,f) : (\forall n)(x \notin \overline{U}_{f\lceil_{n+1}})\}$. Then C is a G_δ–subset of $X \times {}^{\mathbb{N}}\mathbb{N}$. Moreover, for each $x \in X$ the set C_x is closed and nowhere dense. Now apply the fact that X is an $R^{\mathcal{M}}$–set (from $3 \Rightarrow 4$, already established, and Theorem 1): pick $f \in {}^{\mathbb{N}}\mathbb{N} \setminus \bigcup_{x\in X} C_x$. Then the play

$$F(\emptyset), U_{f(1)}, F(U_{f(1)}), U_{f(1),f(2)}, F(U_{f(1)}, U_{f(1),f(2)}), \dots$$

is lost by ONE. □

A set of real numbers in $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ need not be in $\mathsf{S}_1(\mathcal{K},\mathcal{K})$. The reason for this is that in [1] Galvin and Miller show that Martin's Axiom implies the existence of an uncountable first category set which has property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$, while Lusin sets are of the second category.

$\mathcal{K}_\Omega$–Lindelöf sets of real numbers

A space is $\mathcal{K}_\Omega$–Lindelöf if each element of $\mathcal{K}_\Omega$ has a countable subset in $\mathcal{K}_\Omega$. If a set of real numbers is $\mathcal{K}_\Omega$–Lindelöf, then it is easily seen to be $\mathcal{K}$–Lindelöf, and thus a Lusin set. Answering two questions from an earlier version of this paper, Winfried Just proved in [2] that if there is any Lusin set at all, then there is a Lusin set which is not $\mathcal{K}_\Omega$–Lindelöf.

Problem 1. Could it be that there are Lusin sets, but none is $\mathcal{K}_\Omega$–Lindelöf?

Given some special axioms, one can show that there are uncountable $\mathcal{K}_\Omega$–Lindelöf sets of real numbers. In particular: The axiom $\diamond$ asserts that there is a sequence $(A_\alpha : \alpha < \omega_1)$ such that

- $\diamond$.1 For each α, $A_\alpha \subset \alpha$, and
- $\diamond$.2 For every subset S of ω_1, the set $\{\alpha < \omega_1 : S \cap \alpha = A_\alpha\}$ is stationary.

It is well known that the axiom $\diamond$ is consistent relative to the consistency of classical mathematics and implies but is not equivalent to the Continuum Hypothesis.

Theorem 3 ($\diamond$)**.** *There exists an uncountable $\mathcal{K}_\Omega$–Lindelöf set of real numbers.*

Proof. Let $(A_\alpha : \alpha < \omega_1)$ be a sequence as in $\diamond$. Let $(O_\alpha : \alpha < \omega_1)$ bijectively enumerate all the nonempty open subsets of $\mathbb{R}$. Let $(G_\alpha : \alpha < \omega_1)$ bijectively enumerate all the dense G_δ–subsets of $\mathbb{R}$.

For each $\alpha < \omega_1$ put $\mathcal{S}_\alpha = \{O_\gamma : \gamma \in A_\alpha\}$. Then $(\mathcal{S}_\alpha : \alpha < \omega_1)$ is a $\diamond$ sequence for the family of open subsets of $\mathbb{R}$, in the following sense: For $\mathcal{U}$ a collection of open subsets of $\mathbb{R}$ and for $\alpha < \omega_1$, write $\mathcal{U}\lceil_\alpha$ to denote the set $\{O_\gamma \in \mathcal{U} : \gamma < \alpha\}$. Then for every family $\mathcal{U}$ of nonempty open subsets of $\mathbb{R}$, $\{\alpha < \omega_1 : \mathcal{U}\lceil_\alpha = \mathcal{S}_\alpha\}$ is a stationary set.

We shall recursively choose irrational numbers $x_\alpha,\ \alpha < \omega_1$, for which the set $L = \mathbb{Q} \cup \{x_\alpha : \alpha < \omega_1\}$ is the desired Lusin set by recursively choosing for each $\gamma < \omega_1$ $(S_\gamma(\delta) : \delta < \omega_1)$, $(U_n^\gamma : n < \omega)$ and x_γ such that:

- a If $\delta < \gamma < \omega_1$, then $S_\gamma(\delta) = \omega$;
- b If $\gamma \le \alpha < \nu < \omega_1$, then $S_\gamma(\alpha),\ S_\gamma(\nu) \subseteq S_\gamma(\gamma)$ are infinite subsets of ω such that $S_\gamma(\nu) \subseteq^* S_\gamma(\alpha)$;
- c If for the set $\mathbb{Q} \cup \{x_\delta : \delta < \gamma\}$ we have $\mathcal{S}_\gamma$ in $\mathcal{K}_\Omega$, then $U_n^\gamma, n < \omega$ are elements of $\mathcal{S}_\gamma$ such that:
  1. For each finite subset G of $\mathbb{Q} \cup \{x_\delta : \delta < \gamma\}$, for all but finitely many n, $G \subseteq \overline{U_n^\gamma}$, and
  2. for all $\alpha \ge \gamma$, $\{n : x_\alpha \in U_n^\gamma\} \supseteq S_\gamma(\alpha)$;

d If $\mathcal{S}_\gamma$ is *not* in $\mathcal{K}_\Omega$ for $\mathbb{Q} \cup \{x_\delta : \delta < \gamma\}$, then for each n $U_n^\gamma = \mathbb{R}$, and for all $\delta < \omega_1$, $S_\gamma(\delta) = \omega$;

e For $\gamma \leq \delta < \omega_1$, x_δ is a member of $G_\gamma \setminus \{x_\nu : \nu < \delta\}$.

Assuming that this recursive construction can be carried out, L would have the desired properties: By e it would be a Lusin set. To see that it would also be $\mathcal{K}_\Omega$–Lindelöf, let $\mathcal{U}$ be an uncountable element of $\mathcal{K}_\Omega$ for L. Define $f : \omega_1 \to \omega_1$ so that for each α, $f(\alpha) \geq \alpha$ is such that $\mathcal{U}\lceil_{f(\alpha)}$ is in $\mathcal{K}_\Omega$ for $\mathbb{Q} \cup \{x_\delta : \delta < \alpha\}$. Then let C be a closed, unbounded subset of ω_1 such that for all $\alpha \in C$, if $\gamma < \alpha$, then $f(\gamma) < \alpha$. Since $S = \{\alpha < \omega_1 : \mathcal{U}\lceil_\alpha = \mathcal{S}_\alpha\}$ is a stationary set, let ρ be a limit ordinal in $C \cap S$. By the definition of C we see that $\mathcal{U}\lceil_\rho$ is in $\mathcal{K}_\Omega$ for $\mathbb{Q} \cup \{x_\delta : \delta < \rho\}$, and so $\mathcal{S}_\rho$ is in $\mathcal{K}_\Omega$ for this set. By c above, $U_n^\rho,\ n < \omega$ are members of $\mathcal{S}_\rho$ such that each finite subset of $\mathbb{Q} \cup \{x_\gamma : \gamma < \rho\}$ is in all but finitely many of the sets $\overline{U_n^\rho}$. The remaining clause of c implies that $\mathcal{S}_\rho$ is a countable subset of $\mathcal{U}$ which is in $\mathcal{K}_\Omega$ for L.

It remains to show that the recursive construction can be carried out. First, set $S_\gamma(\delta) = \omega$ whenever $\delta < \gamma < \omega_1$.

Now begin the construction by first considering $\mathcal{S}_0$. It is an empty set and so is not in $\mathcal{K}_\Omega$ for $\mathbb{Q}$. Set $U_n^0 = \mathbb{R}$ for each n, $S_0(\delta) = \omega$ for each δ, and choose $x_0 \in G_0 \setminus \mathbb{Q}$. This defines $S_0(0)$, $(U_n^0 : n < \omega)$ and x_0 so that all the relevant requirements for the recursive construction are met.

Let $0 < \alpha < \omega_1$ be given, and assume that for each $\beta < \alpha$ we have defined $(S_\beta(\gamma) : \gamma < \alpha)$, $(U_n^\beta : n < \omega)$ and x_β such that all the relevant requirements of the recursive construction have been met. Now consider $\mathcal{S}_\alpha$. For each $\beta < \alpha$ choose an infinite set T_β such that for all $\gamma < \alpha$ $T_\beta \subseteq^* S_\beta(\gamma)$, and when possible, $T_\beta = \omega$.

If $\mathcal{S}_\alpha$ is not in $\mathcal{K}_\Omega$ for $\mathbb{Q} \cup \{x_\gamma : \gamma < \alpha\}$, then for each n put $U_n^\alpha = \mathbb{R}$, and for each δ put $S_\alpha(\delta) = \omega = T_\alpha$. For each $\beta \leq \alpha$, define

$$H_\beta = \bigcap_{m<\omega} \left(\bigcup_{n \geq m, n \in T_\beta} U_n^\beta \right).$$

Each H_β is a dense G_δ–set. Choose

$$x_\alpha \in \left(\bigcap_{\gamma \leq \alpha} H_\gamma \cap G_\gamma \right) \setminus (\mathbb{Q} \cup \{x_\delta : \delta < \alpha\}),$$

and after this define for each $\gamma < \alpha$, $S_\gamma(\alpha) = \{n \in T_\gamma : x_\alpha \in U_n^\gamma\}$. This specifies x_α, $S_\gamma(\alpha), \gamma \leq \alpha$, and $U_n^\alpha,\ n < \omega$, such that all the requirements of the recursive construction are met.

If $\mathcal{S}_\alpha$ is in $\mathcal{K}_\Omega$ for $\mathbb{Q} \cup \{x_\gamma : \gamma < \alpha\}$, then write this latter set as $\bigcup_{n<\omega} F_n$ where for each n, $\emptyset \neq F_n \subset F_{n+1}$ and F_n is finite. Then choose for each n a $U_n^\alpha \in \mathcal{S}_\alpha$ such that $F_n \subseteq \overline{U_n^\alpha}$. Also set $T_\alpha = \omega$. For each $\beta \leq \alpha$ the set $H_\beta = \bigcap_{m<\omega}(\bigcup_{n \geq m, n \in T_\beta} U_n^\beta)$ is a dense G_δ–subset of $\mathbb{R}$. Then choose

$$x_\alpha \in \left(\bigcap_{\beta \leq \alpha} H_\beta \cap G_\beta \right) \setminus (\mathbb{Q} \cup \{x_\beta : \beta < \alpha\}).$$

After this, define for each $\beta \leq \alpha$: $S_\beta(\alpha) = \{n \in T_\beta : x_\alpha \in U_n^\beta\}$.

This specifies $S_\beta(\alpha), \beta \leq \alpha$, $U_n^\alpha, n < \omega$ and x_α such that all the prescriptions of the recursive construction are met. □

$\mathsf{S}_1(\mathcal{K}_\Omega, \mathcal{K}_\Omega)$

If a space has property $\mathsf{S}_1(\mathcal{K}_\Omega, \mathcal{K}_\Omega)$, then it is $\mathcal{K}_\Omega$–Lindelöf. If a space satisfies $\mathsf{S}_1(\mathcal{K}_\Omega, \mathcal{K}_\Omega)$, then it satisfies $\mathsf{S}_1(\mathcal{K}_\Omega, \mathcal{K})$. Moreover, a space satisfies $\mathsf{S}_1(\mathcal{K}_\Omega, \mathcal{K})$ if, and only if, it satisfies $\mathsf{S}_1(\mathcal{K}, \mathcal{K})$. We know that a Lusin set need not have the selection property $\mathsf{S}_1(\mathcal{K}_\Omega, \mathcal{K}_\Omega)$:

Theorem 4 (CH)**.** *There exists a Lusin set which does not satisfy* $\mathsf{S}_1(\mathcal{K}_\Omega, \mathcal{K}_\Omega)$.

Proof. In [8] we constructed, using the Continuum Hypothesis, a Lusin set L with the property that there is a sequence $(\mathcal{U}_n : n \in \mathbb{N})$ such that each $\mathcal{U}_n$ is a cover of L by clopen sets and each finite subset of L is contained in an element of $\mathcal{U}_n$, and yet for every sequence $(U_n : n \in \mathbb{N})$ with for each n $U_n \in \mathcal{U}_n$, there is a two-element subset F of L such that for each n, $F \not\subseteq U_n$. $\square$

Problem 2. Could it be that there are Lusin sets, but none satisfy $\mathsf{S}_1(\mathcal{K}_\Omega, \mathcal{K}_\Omega)$?

Problem 3. Could it be that some Lusin set is $\mathcal{K}_\Omega$–Lindelöf, but none has property $\mathsf{S}_1(\mathcal{K}_\Omega, \mathcal{K}_\Omega)$?

Using $\diamond$ one can construct Lusin sets which have property $\mathsf{S}_1(\mathcal{K}_\Omega, \mathcal{K}_\Omega)$:

Theorem 5 ($\diamond$)**.** *There exists a Lusin set which has property* $\mathsf{S}_1(\mathcal{K}_\Omega, \mathcal{K}_\Omega)$.

Proof. Observe that $\diamond$ implies: There is a sequence $((A^n_\alpha : n < \omega) : \alpha < \omega_1)$ such that:

1. For each α and for each n, $A^n_\alpha \subseteq \alpha$;
2. Whenever $(A^n : n < \omega)$ is a sequence of subsets of ω_1, then $\{\alpha < \omega_1 : (\forall n)(A^n \cap \alpha = A^n_\alpha)\}$ is a stationary set.

To see this, let Ψ be a bijection from ω_1 to $\omega_1 \times \omega$ such that for all α, if $\Psi(\alpha) = (\gamma, n)$, then $\gamma \leq \alpha$. Let $(B_\alpha : \alpha < \omega_1)$ be a $\diamond$–sequence for ω_1. For each α put $A_\alpha = \{\Psi(\gamma) : \gamma \in B_\alpha\}$. Then for each α and each n put

$$A^n_\alpha = \{\gamma : (\gamma, n) \in A_\alpha\}.$$

From now on, let $((A^n_\alpha : n < \omega) : \alpha < \omega_1)$ be as above. Also, let $(U_\alpha : \alpha < \omega_1)$ bijectively list all the nonempty open subsets of $\mathbb{R}$. For each n and α define

$$\mathcal{S}^n_\alpha = \{U_\gamma : \gamma \in A^n_\alpha\}.$$

Then $((\mathcal{S}^n_\alpha : n < \omega) : \alpha < \omega_1)$ has the property that if $(\mathcal{U}_n : n < \omega)$ is any sequence of families of open subsets of $\mathbb{R}$, and if as before we write for each n and α, $\mathcal{U}_n\lceil_\alpha = \{U_\gamma : \gamma < \alpha \text{ and } U_\gamma \in \mathcal{U}_n\}$, then the set $\{\alpha < \omega_1 : (\forall n)(\mathcal{U}_n\lceil_\alpha = \mathcal{S}^n_\alpha)\}$ is a stationary subset of ω_1.

Let $(G_\alpha : \alpha < \omega_1)$ be a bijective listing of all the dense G_δ–subsets of $\mathbb{R}$.

Since the construction is analogous to that of a $\mathcal{K}_\Omega$–Lindelöf set, we just state the prescriptions for the recursion, trusting that the interested reader will check details as needed. We recursively choose x_α, $\alpha < \omega_1$, such that the set $L = \mathbb{Q} \cup \{x_\alpha : \alpha < \omega_1\}$ has the required properties, by recursively choosing for each $\gamma < \omega_1$ $(S_\gamma(\delta) : \delta < \omega_1)$, $(U^\gamma_n : n < \omega)$ and x_γ such that:

a If $\delta < \gamma < \omega_1$, then $S_\gamma(\delta) = \omega$;
b If $\gamma \leq \alpha < \nu < \omega_1$, then $S_\gamma(\alpha)$, $S_\gamma(\nu) \subseteq S_\gamma(\gamma)$ and $S_\gamma(\nu) \subseteq^* S_\gamma(\alpha)$ are infinite;
c If $(\mathcal{S}^n_\gamma : n < \omega)$ is a sequence from $\mathcal{K}_\Omega$ for $\mathbb{Q} \cup \{x_\delta : \delta < \gamma\}$, then
    1. for each n, $U^\gamma_n \in \mathcal{S}^n_\gamma$,

2. for every finite set $G \subseteq \mathbb{Q} \cup \{x_\delta : \delta < \gamma\}$, for all but finitely many n, $G \subseteq \overline{U_n^\gamma}$, and
3. for all $\alpha \geq \gamma$, $\{n : x_\alpha \in U_n^\gamma\} \supseteq S_\gamma(\alpha)$;

d If some $\mathcal{S}_\gamma^n$ is not in $\mathcal{K}_\Omega$ for $\mathbb{Q} \cup \{x_\delta : \delta < \gamma\}$, then for each n $U_n^\gamma = \mathbb{R}$ and for all δ, $S_\gamma(\delta) = \omega$;

e If $\gamma \leq \delta < \omega_1$, then $x_\delta \in G_\gamma \setminus (\mathbb{Q} \cup \{x_\nu : \nu < \delta\})$. ☐

Ramsey theory

In the following theorem, the symbol $\mathcal{A} \to (\mathcal{B})^n_m$ means: For each $A \in \mathcal{A}$ and for each function $f : [A]^n \to \{1, \ldots, m\}$, there is a $B \in \mathcal{B}$ such that $B \subseteq A$, and f is constant on $[B]^n$.

Theorem 6. *If $X \subseteq \mathbb{R}$ is uncountable and $\mathcal{K}_\Omega \to (\mathcal{K})^2_2$, then X is a Lusin set.*

Proof. Let X be as in the hypotheses. If we can show that every element of $\mathcal{K}_\Omega$ has a countable subset in $\mathcal{K}$, then it follows that every element of $\mathcal{K}$ has a countable subset in $\mathcal{K}$, and this would imply that X is a Lusin set. Since X is second countable, the elements of $\mathcal{K}_\Omega$ have cardinality at most $2^{\aleph_0}$. The well–known negative partition relation $2^{\aleph_0} \not\to (\aleph_1)^2_2$ of Kurepa and Sierpiński provides for each element $\mathcal{U}$ of $\mathcal{K}_\Omega$ a function $f : [\mathcal{U}]^2 \to \{1, 2\}$ which has no uncountable homogeneous set. Thus, the partition relation $\mathcal{K}_\Omega \to (\mathcal{K})^2_2$ implies that every element of $\mathcal{K}_\Omega$ has a countable subset which is in $\mathcal{K}$. ☐

Theorem 7. *If $X \subseteq \mathbb{R}$ is a $\mathcal{K}_\Omega$–Lindelöf set, then for each n $\mathcal{K}_\Omega \to (\mathcal{K})^2_n$.*

Proof. Since X is $\mathcal{K}_\Omega$–Lindelöf, it is a Lusin set. Let $\mathcal{U} \in \mathcal{K}_\Omega$ as well as a function $f : [\mathcal{U}]^2 \to \{1, \ldots, n\}$ be given. We may assume that $\mathcal{U}$ is countable, and enumerate it bijectively as $(U_n : n < \omega)$. Recursively construct sequences $(i_n : n < \omega)$ and $(\mathcal{U}_n : n < \omega)$ so that for each m:

1. $i_m \in \{1, \ldots, n\}$;
2. $\mathcal{U}_{m+1} = \{U_j \in \mathcal{U}_m : j > m + 1 \text{ and } f(\{U_{m+1}, U_j\}) = i_{m+1}\}$ is in $\mathcal{K}_\Omega$;
3. $\mathcal{U}_0 = \{U_j \in \mathcal{U} : j > 1 \text{ and } f(\{U_0, U_j\}) = i_0\}$ is in $\mathcal{K}_\Omega$.

For $j \leq n$, put $\mathcal{V}_j = \{U_n : i_n = j\}$. Then the $\mathcal{V}_j$'s partition each $\mathcal{U}_m$ into finitely many classes. As each $\mathcal{U}_m$ is in $\mathcal{K}_\Omega$, for each m there is a j_m with $\mathcal{V}_{j_m} \cap \mathcal{U}_m$ in $\mathcal{K}_\Omega$. Since for each m we have $\mathcal{U}_{m+1} \subseteq \mathcal{U}_m$, we may fix a specific value j such that for each m, $\mathcal{U}_m \cap \mathcal{V}_j \in \mathcal{K}_\Omega$.

Assign a strategy σ to ONE of the game $\mathsf{G}_1(\mathcal{K}, \mathcal{K})$ as follows: ONE's first move is $\sigma(\emptyset) = \mathcal{U}_0 \cap \mathcal{V}_j$. If TWO responds with $U_{n_0} \in \mathcal{U}_0 \cap \mathcal{V}_j$, then ONE plays $\sigma(U_{n_0}) = \{U_m \in \mathcal{U}_{n_0} \cap \mathcal{V}_j : m > n_0\}$. If TWO now responds with U_{n_1}, then ONE plays $\sigma(U_{n_0}, U_{n_1}) = \{U_m \in \mathcal{U}_{n_1} \cap \mathcal{V}_j : m > n_1\}$, and so on.

Since X is a Lusin set, this strategy is not winning for ONE. Consider a play lost by ONE, say $\sigma(\emptyset), U_{n_0}, \sigma(U_{n_0}), U_{n_1}, \sigma(U_{n_0}, U_{n_1}), \ldots$. Then $\{U_{n_0}, U_{n_1}, \ldots\}$ is in $\mathcal{K}$, and f has value j for each pair from this set. ☐

Problem 4. Could there be a Lusin set which does not satisfy the partition relation $\mathcal{K}_\Omega \to (\mathcal{K})^2_2$?

Problem 5. Could there be a Lusin set which satisfies $\mathcal{K}_\Omega \to (\mathcal{K})^2_2$, but which is not $\mathcal{K}_\Omega$–Lindelöf?

References

1. F. Galvin and A.W. Miller, *γ-sets and other singular sets of real numbers*, **Topology and its Applications** 17 (1984), 145 – 155. MR **85f:**54011
2. W. Just, *More on Lusin sets*, a TeX-file identified by Just as "version of 11/08/96 lusin3.tex".
3. W. Just, A.W. Miller, M. Scheepers and P.J. Szeptycki, *Combinatorics of open covers (II)*, **Topology and its Applications** 73 (1996), 241 – 266. CMP 97:04
4. K. Kunen, *Random and Cohen reals*, **Handbook of Set Theoretic Topology**, North-Holland (1984), 887 – 911. MR **86d:**03049
5. N. Lusin, *Sur un problème de M. Baire*, **Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris** 158 (1914), 1258 – 1261.
6. I. Recław, *Every Lusin set is undetermined in Point-open game*, **Fundamenta Mathematicae** 144 (1994), 43 – 54. MR **95f:**04005
7. F. Rothberger, *Eine Verschärfung der Eigenschaft C*, **Fundamenta Mathematicae** 30 (1938), 50 – 55.
8. M. Scheepers, *Rothberger's property and partition relations*, **The Journal of Symbolic Logic**, **62** (1997), 976–980.
9. S. Willard, *General Topology*, **Addison-Wesley Publishing Company** (1970). MR **41:**9173

Department of Mathematics, Boise State University, Boise, Idaho 83725
E-mail address: `marion@math.idbsu.edu`