

ON ALMOST REPRESENTATIONS OF GROUPS

VALERII FAIZIEV

(Communicated by Dale E. Alspach)

ABSTRACT. We say that a group G belongs to the class \mathcal{K} if every nonunit quotient group of G has an element of order two.

Let H be a Hilbert space and let $U(H)$ be its group of unitary operators. Suppose that groups A and B belong to the class \mathcal{K} and the order of B is more than two. Then the free product $G = A * B$ has the following property. For any $\varepsilon > 0$ there exists a mapping $T : G \rightarrow U(H)$ satisfying the following conditions :

- 1) $\|T(xy) - T(x) \cdot T(y)\| \leq \varepsilon, \quad \forall x, \forall y \in G;$
- 2) for any representation $\pi : G \rightarrow U(H)$ the relation
$$\sup\{\|T(x) - \pi(x)\|, x \in G\} = 2$$

holds.

The question “if we replace a given functional equation by a functional inequality, then under what conditions can we state that the solutions of the inequality are close to the solutions of the equation?” was posed for the functional equation $f(xy) = f(x) \cdot f(y)$ for x, y in the group G in [11] in connection with the results of the papers [6], [7], [8].

For a mapping f of the group G into a semigroup of linear transformations of a vector space, the problem on sufficient conditions of the coincidence of the solution of a functional inequality $\|f(xy) - f(x) \cdot f(y)\| < c$ with a solution of the corresponding functional equation $f(xy) - f(x) \cdot f(y) = 0$ was studied in [1], [10]. In the papers [4], [5], [9] it was independently shown that if a continuous mapping f of a compact group G into the algebra of endomorphisms of a Banach space satisfies the relation $\|f(xy) - f(x) \cdot f(y)\| < \delta$ for all $x, y \in G$ with a sufficiently small $\delta > 0$, then it is ε -close to a continuous representation g of the same group in the same Banach space (i.e., we have $\|f(x) - g(x)\| < \varepsilon$ for all $x \in G$).

Let H be a Hilbert space and let $U(H)$ be the group of unitary operators of H endowed by operator-norm topology. If H is n -dimensional $n \in \mathbb{N}$, the group $U(H)$ is denoted by $U(n)$.

Definition. Let $0 < \varepsilon < 2$. Let T be a mapping of a group G into $U(H)$. We say that T is an ε -representation if for any x, y from group G the relation

$$\|T(xy) - T(x)T(y)\| < \varepsilon$$

holds.

Received by the editors November 25, 1996 and, in revised form, May 13, 1997.

1991 *Mathematics Subject Classification.* Primary 20C99.

Key words and phrases. Representation, ε -representation, pseudocharacter.

V. Milman raised the following question: Let $\rho : G \rightarrow U(H)$ be an ε -representation with small ε . Is it true that ρ is near to an actual representation π of the group G in H , i.e., does there exist some small $\delta > 0$ such that $\|\rho(x) - \pi(x)\| < \delta$ for all $x \in G$?

Answering this question Kazhdan [9] obtained the following:

Theorem. *There is a group Γ with the following property. For any $0 < \varepsilon < 1$ and any natural number $n > \frac{3}{\varepsilon}$ there exists an ε -representation ρ such that for any homomorphism $\pi : G \rightarrow U(n)$ the relation*

$$\|\rho - \pi\| = \sup\{\|\rho(x) - \pi(x)\|; \quad x \in \Gamma\} > \frac{1}{10}$$

holds.

Note that the group Γ has the following presentation in terms of generations and defining relations: $\Gamma = \langle x, y, a, b \mid x^{-1}y^{-1}xy a^{-1}b^{-1}ab \rangle$. The aim of this article is to strengthen this result. Namely we establish the following:

Theorem 1. *There exists a group C such that for any $0 < \varepsilon < 2$ and any Hilbert space H there exists an ε -representation t of the group C in H satisfying the following condition: for any representation $\pi : C \rightarrow U(H)$ the relation $\|t - \pi\| = 2$ is valid.*

Definition. A *pseudocharacter* of a group M is a real-valued function φ on M such that:

- 1) the set $\{\varphi(xy) - \varphi(x) - \varphi(y) \mid x, y \in M\}$ is bounded;
- 2) $\varphi(x^n) = n\varphi(x)$ for any integer n and any x from M .

We say that a pseudocharacter φ is nontrivial if it is not an additive real character of the group M . It is easy to see that property 2) from the definition of a pseudocharacter implies that $\varphi(a) = 0$ for every element a of finite order.

We say that a group M belongs to the class \mathcal{K} if every nonunit quotient group of M has an element of order two. Note that not only 2-groups belong to the class \mathcal{K} . Indeed, it is easy to verify that for any prime number p the group $\langle a, b \mid a^2 = b^p = 1, a^{-1}ba = b^{-1} \rangle$ belongs to \mathcal{K} . It is evident that the class \mathcal{K} is closed under free and direct products.

Lemma 1. *Let G be a group from the class \mathcal{K} and let f be a nontrivial pseudocharacter of G such that the range of the function f belongs to the set of rationals Q . Let H be an arbitrary Hilbert space and let E be an identity operator of H . Then the mapping*

$$\Phi : x \rightarrow E_f(x) = \exp(if(x)) \cdot E$$

satisfies the following condition: For every unitary representation π of the group G in H the relation $\|\Phi - \pi\| = 2$ holds.

Proof. Suppose that $\pi(x) = E \quad \forall x \in G$. Let us verify that $\|\Phi - E\| = 2$. Let g be an element from G such that $f(g) \neq 0$. Then there exists a natural number k such that $f(g^k) \in Z \setminus \{0\}$. Hence, the range of the function $n \rightarrow \exp(if(g^{k \cdot n}))$, $n \in Z$, is everywhere dense in the unit circle and we obtain the relation

$$\begin{aligned} \|\Phi - E\| &= \sup\{\|\Phi(x) - E\|; \quad x \in G\} \geq \sup\{\|\Phi(g^{k \cdot n}) - E\|; \quad n \in Z\} \\ &= \sup\{|\exp(i \cdot n \cdot f(g^k)) - 1|; \quad n \in Z\} = 2. \end{aligned}$$

Suppose that $\pi(x) \neq E$ for some $x \in G$. The definition of the class \mathcal{K} implies that there exists $a \in G$ such that $\pi(a) = A \neq E$ and $A^2 = E$. Then either $\Phi(a) = E$ or the subgroup generated by $\exp(i \cdot f(a^2))$ is dense in the unit circle. In either case, it is not difficult to show that $\|\Phi - E\| = 2$.

The lemma is proved. \square

Denote by $G = A * B$ the free product of nonunit groups A and B . Let $A_0 = A \setminus 1$, $B_0 = B \setminus 1$.

Definition. Let g be a nonunit element from the group G . By the *canonical form* of an element g we mean its presentation in the form

$$(1) \quad g = c_1 c_2 \cdots c_k,$$

where $c_i \in A_0 \cup B_0$ and $c_i c_{i+1} \notin A \cup B$.

And we put $\dot{g} = c_1$, $\ddot{g} = c_k$, $|g| = k$. If $g = 1$ we put $\dot{g} = \ddot{g} = 1$, $|g| = 0$.

Definition. Let $g \neq 1$ and (1) be its canonical form. By a *subdivision* of g we mean its presentation in the form

$$g = g_1 g_2 \cdots g_m,$$

where g_i , $i = 1, 2, \dots, m$ are canonical forms and $\ddot{g}_i \cdot \dot{g}_{i+1} \notin A \cup B$.

Now assume that the group B contains an element b_0 of order more than two. Let us define the function f as follows: $f(b_0) = 1$, $f(b_0^{-1}) = -1$ and $f(c) = 0$ if $c \in A \cup B \setminus \{b_0, b_0^{-1}\}$.

Now extend f to the group G by setting

$$f(g) = \sum_{i=1}^k f(c_i),$$

where g has canonical form (1).

It is evident that for every g from the group G the equality $f(g^{-1}) = -f(g)$ holds. Note that if $g = g_1 g_2 \cdots g_m$ is a subdivision of g , then

$$f(g) = \sum_{i=1}^m f(g_i).$$

Denote by $\Delta(u, v)$ the number $f(uv) - f(u) - f(v)$.

Lemma 2. For any u, v from G the relation

$$(2) \quad |\Delta(u, v)| \leq 2$$

is valid.

Proof. It is clear if $|u| \leq 1$ or $|v| \leq 1$, then the relation $|\Delta(u, v)| \leq 2$ holds.

Now we assume that $|u| \geq 1$, $|v| \geq 1$. It is obvious that if $|\ddot{u} \cdot \dot{v}| = 2$, then $\Delta(u, v) = 0$.

Let $|\ddot{u} \cdot \dot{v}| = 1$. Then either a) $\ddot{u} \cdot \dot{v}, \ddot{u}, \dot{v} \in A$ or b) $\ddot{u} \cdot \dot{v}, \ddot{u}, \dot{v} \in B$.

In the first case we obtain subdivisions $u = u_1 \cdot a$, $v = \alpha \cdot v_1$, where $a, \alpha \in A$; $a\alpha \neq 1$. Hence $\Delta(u, v) = \Delta(a, \alpha) = 0$.

In case b) we obtain subdivisions $u = u_1 \cdot b$, $v = \beta \cdot v_1$, where $b, \beta \in B$ and $b\beta \neq 1$. Hence $|\Delta(u, v)| = |\Delta(b, \beta)| \leq 2$.

Now consider the case $|\ddot{u} \cdot \dot{v}| = 0$. There are four subcases:

- 1) $u = u_1 \cdot t, v = t^{-1} \cdot v_1, |\ddot{u}_1 \dot{v}_1| = 1$;
- 2) $u = t, v = t^{-1} \cdot v_1$;
- 3) $u = u_1 \cdot t, v = t^{-1}$;
- 4) $uv = 1$

where $u = u_1 \cdot t, v = t^{-1} \cdot v_1$ are subdivisions.

In the first case we have $|\Delta(u, v)| = |\Delta(u_1, v_1)| \leq 2$. In the other cases we have $\Delta(u, v) = 0$.

The lemma is proved. \square

Now we define the function φ by setting

$$\varphi(x) = \lim_{n \rightarrow \infty} \frac{1}{2^n} f(x^{2^n}).$$

From [3] we know that the function φ is well-defined and is a pseudocharacter of the group G , and for arbitrary elements x, y from the group G the relations

$$|\varphi(x) - f(x)| \leq 2, \quad |\varphi(xy) - \varphi(x) - \varphi(y)| \leq 6$$

hold.

Let us verify that φ is integer-valued. Lemma 2 from [2] implies that every pseudocharacter of an arbitrary group is constant on the conjugacy classes. It is evident that $\varphi|_{A \cup B} \equiv 0$. Let D be the free subsemigroup of G , generated by the set $X = \{ab | a \in A_0, b \in B_0\}$. It is clear that the function $f|_D$ is an additive character of the semigroup D . This fact implies that $\varphi|_D = f|_D$. Indeed, let $v = a_1 b_1 \cdots a_k b_k \in D$, then it is evident that $v^m = \underbrace{v \cdot v \cdots v}_m$ is a subdivision of v^m for any m from N . Hence, $f(v^{2^n}) = 2^n f(v)$. Now from the definition of the function φ we obtain that $\varphi(v) = f(v)$. Since every element of the group G is conjugate to some element from the set $D \cup A \cup B$ we conclude that the function φ is integer-valued.

Poof of Theorem 1. Suppose that groups A and B belong to the class \mathcal{K} and assume B has an element of order more than two. Then the group $C = A * B$ has a nontrivial integer-valued pseudocharacter φ . For each $m \in N$ we put $\varphi_m = \frac{1}{m} \varphi$. Then for any elements x, y from the group G we have the relation

$$|\varphi_m(xy) - \varphi_m(x) - \varphi_m(y)| \leq \frac{1}{m} \cdot 6.$$

Let

$$\Phi_m(x) = \exp(i\varphi_m(x)) \cdot E.$$

From Lemma 1 we obtain that for any Hilbert space H and every representation π of the group C in H the relation

$$\|\Phi_m - \pi\| = 2$$

holds.

Furthermore, we have

$$\begin{aligned} \|\Phi_m(xy) - \Phi_m(x)\Phi_m(y)\| &= \|\exp(i\varphi_m(xy)) - \exp(i(\varphi_m(x) + \varphi_m(y)))\| \\ &= |\exp(i(\varphi_m(xy) - \varphi_m(x) - \varphi_m(y))) - 1|. \end{aligned}$$

And we obtain that for every $\varepsilon > 0$ there exists $l = l(\varepsilon)$ such that if $m > l$, then the relation

$$\|\Phi_m(xy) - \Phi_m(x)\Phi_m(y)\| < \varepsilon$$

holds.

This completes the proof of Theorem 1. \square

ACKNOWLEDGEMENT

I would like to express my thanks to the referee who corrected many mistakes in a previous version of the manuscript.

REFERENCES

1. Baker, J., Lawrence, L., Zorzitto, F., *The stability of the equation $f(x+y) = f(x)f(y)$* , Proc. Amer. Math. Soc. **74** (2) (1979), 242–246. MR **80d**:39009
2. Faiziev, V.A., *On the spaces of pseudocharacters on free product of semigroups*, Mat. Zametki **52** (1992), 119–130. MR **93m**:20087
3. Faiziev, V.A., *Pseudocharacters on semidirect product of semigroups*, Mat. Zametki **53** (2) (1993), 132–139. MR **94h**:20070
4. Grove, K., Karcher, H., Ruh, E. A., *Jacobi fields and Finsler metrics on compact Lie groups with an application to differential pinching problems*, Math. Ann. **211** (1) (1974), 7–21. MR **50**:8391
5. de la Harpe, P., Karoubi, M., *Représentations approchées d'un groupe dans une algèbre de Banach*, Manuscripta Math. **22** (3) (1977), 297–310. MR **58**:16948
6. Hyers, D. H., *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA **27** (2) (1941), 222–224. MR **2**:315a
7. Hyers, D. H., Ulam, S. M., *On approximate isometry*, Bull. Amer. Math. Soc. **51** (1945), 228–292. MR **7**:123f
8. Hyers, D. H., Ulam, S. M., *Approximate isometry on the space of continuous functions*, Ann. of Math. **48** (2) (1947), 285–289. MR **8**:588b
9. Kazhdan, D., *On ε -representations*, Israel J. Math. **43** (4) (1982), 315–323. MR **84h**:22011
10. Shtern, A.I., *Stability of homomorphisms into the group R^** , Vestnik Moskov Univ. Ser. I Mat. Mekh. **37** (3) (1982), 33–36. MR **83m**:39008
11. Ulam, S. M., *A Collection of Mathematical Problems*, Wiley & Sons, Interscience Publ; New York –London, 1960. MR **22**:10884

INSTITUTE FOR MATHEMATICS WITH COMPUTATIONAL CENTER, TADZHIKISTAN ACADEMY OF SCIENCES, DUSHANBE, TADZHIKISTAN

Current address: Shirokaia St. 7-3-137, 129282 Moscow, Russia