

## SMITH EQUIVALENCE OF REPRESENTATIONS FOR FINITE PERFECT GROUPS

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ABSTRACT. Using smooth one-fixed-point actions on spheres and a result due to Bob Oliver on the tangent representations at fixed points for smooth group actions on disks, we obtain a similar result for perfect group actions on spheres. For a finite group  $G$ , we compute a certain subgroup  $IO'(G)$  of the representation ring  $RO(G)$ . This allows us to prove that a finite perfect group  $G$  has a smooth 2-proper action on a sphere with isolated fixed points at which the tangent representations of  $G$  are mutually nonisomorphic if and only if  $G$  contains two or more real conjugacy classes of elements not of prime power order. Moreover, by reducing group theoretical computations to number theory, for an integer  $n \geq 1$  and primes  $p, q$ , we prove similar results for the group  $G = A_n$ ,  $SL_2(\mathbb{F}_p)$ , or  $PSL_2(\mathbb{F}_q)$ . In particular,  $G$  has Smith equivalent representations that are not isomorphic if and only if  $n \geq 8$ ,  $p \geq 5$ ,  $q \geq 19$ .

### INTRODUCTION

In 1960, P.A. Smith [Sm] asked the question: if a finite group  $G$  acts smoothly on a sphere  $S^n$  with exactly two fixed points  $x$  and  $y$ , is it true that  $T_x(S^n) \cong T_y(S^n)$ ? The tangent spaces  $T_x(S^n)$  and  $T_y(S^n)$  are considered as the representations of  $G$  determined by the derivative of the  $G$ -action, which we refer to as to the *tangent representations*. In the development of transformation group theory in the subsequent decades, the isomorphism question for the tangent representations has been posed for a smooth action of  $G$  on any homotopy sphere  $\Sigma$  with  $\Sigma^G = \{x, y\}$ .

Atiyah–Bott [AB] and Milnor [Mi] answered the isomorphism question affirmatively for  $G = \mathbb{Z}_p$ , the cyclic group of prime order  $p$ , as well as for any finite group  $G$  whose action on  $\Sigma$  is free outside of  $x$  and  $y$ . Without this assumption, Sanchez [Sa] proves that  $T_x(\Sigma) \cong T_y(\Sigma)$  when  $G$  is a finite group of odd order and for each subgroup  $H$  of  $G$ , the  $H$ -fixed point set  $\Sigma^H$  is connected or  $\Sigma^H = \{x, y\}$ . By Smith theory, this condition on  $\Sigma^H$  holds when  $G$  is a finite  $p$ -group for a prime  $p$ , and thus,  $T_x(\Sigma) \cong T_y(\Sigma)$  for  $p$  odd. For  $p = 2$ , there is only a weaker 2-divisibility relation between  $T_x(\Sigma)$  and  $T_y(\Sigma)$  which goes back to Bredon [Br].

In the early eighties, Cappell and Shaneson [CS1], [CS2] answered the isomorphism question negatively in the case of smooth actions of  $G$  on  $S^n$  with  $n \geq 9$  ( $n$  odd) for  $G = \mathbb{Z}_{2^a q}$  with  $a \geq 2$  and  $q \geq 3$  ( $q$  odd), as well as in the case of smooth

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actions of  $G$  on 9-dimensional homotopy spheres for  $G = \mathbb{Z}_{4q}$  with  $q \geq 2$ . It follows easily from the Atiyah–Bott–Milnor theorem and elementary representation character arguments that the smallest group  $G$  producing a negative answer to the isomorphism question is  $G = \mathbb{Z}_8$ .

In the late seventies and early eighties, Petrie [Pe1], [Pe2], [Pe3] announced his program of developing equivariant surgery and using it to construct smooth actions on homotopy spheres. Following Petrie’s definition, we say that two representations  $U$  and  $V$  of  $G$  are *Smith equivalent* if there exists a smooth action of  $G$  on a homotopy sphere  $\Sigma$  such that  $\Sigma^G = \{x, y\}$  and, as representations of  $G$ ,  $U \cong T_x(\Sigma)$  and  $V \cong T_y(\Sigma)$ . By using equivariant surgery, Petrie proved that for each finite abelian group  $G$  of odd order with at least four noncyclic Sylow subgroups, there exist nonisomorphic Smith equivalent representations  $U$  and  $V$  of  $G$  (see [PR] for a detailed proof). Then, his students and collaborators obtained similar results for different classes of finite groups. In particular, the articles [DP], [DW], [DS] contain a lot of information on the order of a cyclic group  $G$  which has nonisomorphic Smith equivalent representations. We refer the reader to the survey articles [CS3], [DPS], [MP] for more history on related results.

In this paper, a smooth action of a finite group  $G$  on a homotopy sphere  $\Sigma$  is called *2-proper* if for each element  $g \in G$  of order  $2^a$  with  $a \geq 3$ ,  $\Sigma^g$  is connected. If  $\Sigma^G = \{x, y\}$ , this amounts to saying (by Smith theory) that  $\Sigma^g$  contains  $\Sigma^G$  as a proper subset.

In accordance with Petrie’s definition, two representations  $U$  and  $V$  of  $G$  are called *2-proper Smith equivalent* if there exists a smooth 2-proper action of  $G$  on a homotopy sphere  $\Sigma$  such that  $\Sigma^G = \{x, y\}$  and, as representations of  $G$ ,  $U \cong T_x(\Sigma)$  and  $V \cong T_y(\Sigma)$ .

Now, we state the main results of this paper (Theorems A and B). Hereafter, for a finite group  $G$ , we denote by  $r_G$  the number of real conjugacy classes of elements of  $G$  not of prime power order.

**Theorem A.** *A finite perfect group  $G$  has nonisomorphic 2-proper Smith equivalent representations if and only if  $r_G \geq 2$ . A finite perfect group  $G$  has a smooth action on a sphere with any given number  $k \geq 3$  of fixed points at which the tangent representations are mutually nonisomorphic if and only if  $r_G \geq 2$ .*

Recall that each finite simple nonabelian group is perfect. For an integer  $n \geq 1$ , the alternating group  $A_n$  is simple and nonabelian (equivalently, perfect and nontrivial) if and only if  $n \geq 5$ . Given the field  $\mathbb{F}_p$  of  $p$  elements for a prime  $p$ , the special linear group  $\mathrm{SL}_2(\mathbb{F}_p)$  is perfect if and only if  $p \geq 5$ , and the projective special linear group  $\mathrm{PSL}_2(\mathbb{F}_p)$  is simple (equivalently, perfect) if and only if  $p \geq 5$ .

**Theorem B.** *Let  $G = A_n$ ,  $\mathrm{SL}_2(\mathbb{F}_p)$ , or  $\mathrm{PSL}_2(\mathbb{F}_q)$  for an integer  $n \geq 1$  and primes  $p, q$ . Then  $r_G \leq 1$  for  $n \leq 7$ ,  $p \leq 3$ ,  $q \leq 17$ , and  $r_G \geq 2$  for  $n \geq 8$ ,  $p \geq 5$ ,  $q \geq 19$ . Moreover,  $G$  has nonisomorphic 2-proper Smith equivalent representations if and only if  $r_G \geq 2$ . Similarly,  $G$  has nonisomorphic Smith equivalent representations if and only if  $r_G \geq 2$ . Also,  $G$  has a smooth action on a sphere with any given number  $k \geq 3$  of fixed points at which the tangent representations are mutually nonisomorphic if and only if  $r_G \geq 2$ .*

This paper is divided into two parts. In Part 1, using the Deleting–Inserting Theorem [LMP, Thm. 2.2], we show that each finite nontrivial perfect group  $G$  has a smooth action on a sphere with exactly one fixed point at which the tangent

representation is stably isomorphic to a given  $G$ -oriented representation  $V$  of  $G$  with  $\dim V^G = 0$  (Theorem 1.1). Then, using a description due to Oliver [O] of the tangent representations at fixed points for smooth finite group actions on disks (Theorem 1.2), we present a corresponding result for smooth finite perfect group actions on spheres (Theorem 1.3). Here, the idea is to take the double of the disk obtained by Oliver's construction, and cancel the extra fixed points by forming connected sums of the doubled disk and spheres equipped with smooth one-fixed-point actions. This allows us to prove that Theorem A holds when the condition  $r_G \geq 2$  is replaced by the condition  $IO'(G) \neq 0$  for a certain subgroup  $IO'(G)$  of the representation ring  $RO(G)$  (Theorem 1.7). Finally, for a finite nontrivial perfect group  $G$ , we describe a subgroup of  $RO(G)$  whose elements all occur as differences of 2-proper Smith equivalent representations of  $G$  (Corollary 1.8).

In Part 2, for a finite group  $G$ , we compute the rank of the free abelian group  $IO'(G)$  in terms of the number  $r_G$ . In particular,  $IO'(G) \neq 0$  if and only if  $r_G \geq 2$  (Lemma 2.1). Theorem A now follows from Theorem 1.7. Using elementary arguments for  $G = A_n$  and number theory for  $G = \mathrm{SL}_2(\mathbb{F}_p)$  or  $\mathrm{PSL}_2(\mathbb{F}_q)$ , we compute  $r_G$  for any  $n \geq 1$  and primes  $p, q$  (Propositions 2.2, 2.3, 2.4). In particular,  $r_G \geq 2$  if and only if  $n \geq 8$ ,  $p \geq 5$ ,  $q \geq 19$ . For  $n \leq 7$ ,  $p \leq 3$ ,  $q \leq 13$ ,  $G$  has no element of order 8 (Propositions 2.2, 2.3, 2.4). With some additional arguments for  $\mathrm{PSL}_2(\mathbb{F}_{17})$  (Example 2.5 and Lemma 2.6), we show that for  $G = A_n$ ,  $\mathrm{SL}_2(\mathbb{F}_p)$ , or  $\mathrm{PSL}_2(\mathbb{F}_q)$ , the following statement is true (Theorem 2.7). If  $r_G \leq 1$ , then for any smooth action of  $G$  on a homotopy sphere, at any two isolated fixed points of the action, the tangent representations of  $G$  are isomorphic. Theorem 2.7 also contains the following two statements. If  $r_G \geq 2$ , then there exist nonisomorphic 2-proper Smith equivalent representations of  $G$ . If  $r_G \geq 2$ , then there exists a smooth action of  $G$  on a standard sphere with any given number  $k \geq 3$  of fixed points at which the tangent representations of  $G$  are mutually nonisomorphic. This proves Theorem B.

## PART 1. TOPOLOGY

In this paper, a real representation  $V$  of  $G$  is called  $G$ -oriented if for each subgroup  $H$  of  $G$ , the  $H$ -fixed point set  $V^H$  is oriented and each element of the normalizer  $N_G(H)$  of  $H$  in  $G$  acts on  $V^H$  via an orientation preserving transformation; cf. [LMP, (2.1.3)]. For a finite nonsolvable group  $G$  and a prescribed representation  $W$  of  $G$ , [LMP] constructs a smooth action of  $G$  on  $S^n$  ( $n = \dim W$ ) with exactly one fixed point  $x$ , such that  $T_x(S^n) \cong W$ . We complement that result for a finite perfect group  $G$ , to the effect that for any real  $G$ -oriented representation  $V$  with  $\dim V^G = 0$  and the prescribed representation  $W$ , the action is such that  $T_x(S^n) \cong V \oplus W$ . This result is a special case of the main result of [LM]. However, we present a proof here to point out that, unlike in [LM], no surgery with middle-dimensional singularities is needed because we are working with the strong gap hypothesis of [LMP]. Hereafter, for a finite perfect group  $G$ , we set  $V(G) = \mathbb{R}[G] - \mathbb{R}$ , where  $\mathbb{R}[G]$  denotes the real regular representation of  $G$  and the subtracted summand  $\mathbb{R}$  has the trivial action of  $G$ .

**Theorem 1.1.** *Let  $G$  be a finite nontrivial perfect group and  $V$  a real  $G$ -oriented representation with  $\dim V^G = 0$ . Then, for any even integer  $\ell \geq \max\{6, \dim V\}$ , there exists a smooth action of  $G$  on  $S^n$  with exactly one fixed point  $x$ , and the action is such that  $T_x(S^n) \cong V \oplus \ell V(G)$ . In particular,  $n = \dim V + \ell(|G| - 1)$ .*

Moreover, for each proper subgroup  $H$  of  $G$ ,  $\dim(S^n)^H \geq 6$  and  $(S^n)^H$  is simply connected.

*Proof.* As  $G$  is perfect, the representation  $\ell V(G)$  satisfies the strong gap hypothesis for the family  $\mathcal{I}(G)$  of all proper subgroups of  $G$ , when  $\ell \geq 5$  [LMP, (5.4)]. It follows that  $V \oplus \ell V(G)$  satisfies the strong gap hypothesis when  $\ell \geq \dim V$  [LMP, Remark to Prop. 5.1]. Moreover, for any even integer  $\ell \geq 2$ ,  $\ell V(G)$  is  $G$ -oriented, as it is the realification of the complex representation  $\ell/2(\mathbb{C}[G] - \mathbb{C})$ . Now, for an even integer  $\ell \geq \max\{6, \dim V\}$ , set  $U = V \oplus \ell V(G)$ . As  $V$  is  $G$ -oriented, so is  $U$ . Moreover,  $U$  satisfies the strong gap hypothesis for  $\mathcal{I}(G)$ . Let  $Y = S(U \oplus \mathbb{R})$  be the  $G$ -invariant sphere of  $U \oplus \mathbb{R}$ , where  $G$  acts trivially on  $\mathbb{R}$ . Then  $Y^G = \{x, y\}$  and, as representations of  $G$ ,  $T_x(Y) \cong T_y(Y) \cong U$ .

Now, we wish to apply the Deleting–Inserting Theorem [LMP, Thm. 2.2] to remove from  $Y^G$  one point, say  $y$ . In order to do it, we have to check that the assumptions (2.1.1)–(2.1.4) of Situation 2.1 [LMP] hold. Clearly,  $\text{Iso}(G, Y \setminus Y^G) = \mathcal{I}(G)$ . For each  $H \in \mathcal{I}(G)$ ,  $Y^H$  is a sphere with  $\dim Y^H \geq 6$ , and hence  $Y^H$  is simply connected. Since  $U$  is  $G$ -oriented, thus for each  $H \in \mathcal{I}(G)$ , each element of  $N_G(H)$  acts on  $Y^H$  via an orientation preserving transformation. Finally,  $Y$  satisfies the strong gap hypothesis for  $\mathcal{I}(G)$  because  $U$  does. The Deleting–Inserting Theorem [LMP, Thm. 2.2] now produces a smooth action of  $G$  on a homotopy sphere  $X$  with  $X^G = \{x\}$  and  $T_x(X) \cong U$ . Moreover, for each  $H \in \mathcal{I}(G)$ ,  $\dim X^H \geq 6$  and  $X^H$  is simply connected. It follows from [LMP, Prop. 1.3] that  $X$  can be chosen to be the sphere  $S^n$  with  $n = \dim U$ .  $\square$

For a finite group  $G$ , consider the following subgroups of the real representation ring  $RO(G)$ :

$$IO(G) = \bigcap_p \text{Ker}(RO(G) \rightarrow RO(G_p)), \quad IO'(G) = IO(G) \cap \text{Ker}(RO(G) \xrightarrow{d^G} \mathbb{Z})$$

where the maps  $RO(G) \rightarrow RO(G_p)$  are defined by restricting to Sylow subgroups  $G_p$  of  $G$ , and  $d^G(U - V) = \dim U^G - \dim V^G$  for representations  $U$  and  $V$  of  $G$ . Clearly, the condition that  $U - V \in IO(G)$  amounts to saying that  $U$  and  $V$  are isomorphic when restricted to any Sylow (and thus, any prime power order) subgroup of  $G$ . Now, we restate a theorem which goes back to Bob Oliver [O, Thm. 0.4].

**Theorem 1.2.** *Let  $G$  be a finite group not of prime power order. Let  $V_1, \dots, V_k$  be real representations of  $G$  such that  $\dim V_i^G = 0$  and  $V_i - V_j \in IO(G)$  for all  $1 \leq i, j \leq k$ . Assume  $k \equiv 1 \pmod{n_G}$ . Then, for some real representation  $V$  of  $G$  with  $\dim V^G = 0$ , there exists a smooth action of  $G$  on  $D^n$  with exactly  $k$  fixed points  $x_1, \dots, x_k$ , such that  $T_{x_i}(D^n) \cong V_i \oplus V$  for  $i = 1, \dots, k$ .*

If  $G$  is a finite nontrivial perfect group (or more generally, a finite nonsolvable group), then the integer  $n_G = 1$  (cf. [O, Thm. 0.3]). Thus, for such a group  $G$  in Theorem 1.2, there is no restriction on the number  $k$  of fixed points.

**Theorem 1.3.** *Let  $G$  be a finite nontrivial perfect group. Let  $V_1, \dots, V_k$  be real  $G$ -oriented representations such that  $\dim V_i^G = 0$  and  $V_i - V_j \in IO(G)$  for all  $1 \leq i, j \leq k$ . Then, for some real  $G$ -oriented representation  $W$  with  $\dim W^G = 0$ , there exists a smooth action of  $G$  on  $S^n$  with exactly  $k$  fixed points  $x_1, \dots, x_k$ , such that  $T_{x_i}(S^n) \cong V_i \oplus W$  for  $i = 1, \dots, k$ . Moreover, for each element  $g \in G$  of prime power order,  $(S^n)^g$  is connected.*

*Proof.* By Theorem 1.2, there there exist a real representation  $V$  of  $G$  with  $\dim V^G = 0$  and a smooth action of  $G$  on the disk  $D^m$  of dimension  $m = \dim V_i + \dim V$  such that  $D^m$  contains exactly  $k$  fixed points  $x_1, \dots, x_k$  at which the tangent representations are isomorphic to  $V_1 \oplus V, \dots, V_k \oplus V$ . For an even integer  $\ell \geq \max\{6, \dim(V_i \oplus V \oplus V)\}$ , set  $W = V \oplus V \oplus \ell V(G)$ . Then  $W$  is a real  $G$ -oriented representation with  $\dim W^G = 0$ .

Consider the disk  $D^n = D^m \times D(V \oplus \ell V(G))$  with the diagonal action of  $G$ , and form the double  $\partial(D^n \times D^1) = S^n$  of  $D^n$ , where  $G$  acts trivially on  $D^1$ . The resulting action of  $G$  on  $S^n$  has exactly  $2k$  fixed points  $x_1, y_1, \dots, x_k, y_k$  and, as representations of  $G$ ,  $T_{x_i}(S^n) \cong T_{y_i}(S^n) \cong V_i \oplus W$  for  $i = 1, \dots, k$ . Since  $V_i \oplus V \oplus V$  is  $G$ -oriented, it follows from Theorem 1.1 that there exists a smooth action of  $G$  on a copy  $S_i^n$  of the sphere of dimension  $n = \dim(V_i \oplus W)$  with exactly one fixed point, say  $z_i$ , at which the tangent representation is isomorphic to  $V_i \oplus W$ . To complete the proof, we form the connected sum of  $S^n$  with  $S_i^n$  by piecing together a small neighborhood of  $y_i$  in  $S^n$  with a small neighborhood of  $z_i$  in  $S_i^n$  for  $i = 1, \dots, k$ . It follows from the construction that for each element  $g \in G$  of prime power order,  $\dim(S^n)^g \geq 6$  and, by Smith theory,  $(S^n)^g$  is connected.  $\square$

**Lemma 1.4.** *Let  $G$  be a finite group acting smoothly on a homotopy sphere  $\Sigma$ . Assume  $\Sigma^G = \{x, y\}$ , and set  $U = T_x(\Sigma)$  and  $V = T_y(\Sigma)$ . If the action of  $G$  on  $\Sigma$  is 2-proper, then  $U - V \in IO'(G)$ .*

*Proof.* By the Slice Theorem,  $\dim U^G = \dim V^G = 0$ . As recalled in the introduction, the characters  $\chi_U$  and  $\chi_V$  of  $U$  and  $V$  agree on elements of  $G$  of odd prime power order. They also agree on any element of order 2 or 4, because the characters of representations of cyclic groups of order 2 or 4 can be read off the fixed point set dimensions. Hence, if the action is 2-proper,  $\chi_U$  and  $\chi_V$  agree on all elements of prime power order, which means that  $U - V \in IO(G)$ . Since  $\dim U^G = \dim V^G$ , thus  $U - V \in IO'(G)$ .  $\square$

*Remark 1.5.* Let  $G$  be a finite group acting smoothly on a homotopy sphere  $\Sigma$ . Assume  $\Sigma^G$  contains two points  $x$  and  $y$  which belong to connected components of  $\Sigma^G$  of the same dimension. Set  $U = T_x(\Sigma)$  and  $V = T_y(\Sigma)$ . By the Slice Theorem,  $\dim U^G = \dim V^G$ . Assume further that  $\Sigma^g$  is connected for each element  $g \in G$  of prime power order (and thus, in particular, the action of  $G$  on  $\Sigma$  is 2-proper). Then, it follows that  $U - V \in IO'(G)$ . By Smith theory, the latter assumption holds when  $\Sigma^G$  contains a third point.

**Example 1.6.** For a finite group  $G$ , assume  $IO'(G) \neq 0$  and choose  $U - V \neq 0$  in  $IO'(G)$ . The complexifications of  $U - U^G$  and  $V - V^G$  have the realifications

$$U_0 = (U - U^G) \oplus (U - U^G) \quad \text{and} \quad V_0 = (V - V^G) \oplus (V - V^G).$$

As  $RO(G)$  is a free abelian group,  $2(U - V) \neq 0$ . In  $RO(G)$ ,  $U_0 - V_0 = 2(U - V)$ . Thus,  $U_0$  and  $V_0$  are not isomorphic. For  $i = 1, \dots, k$  with  $k \geq 2$ , set  $V_i = (i-1)U_0 \oplus (k-i)V_0$ . Then, each  $V_i$  is  $G$ -oriented,  $\dim V_i^G = 0$ , and  $V_i - V_j \in IO(G)$ . Clearly,  $V_i$  and  $V_j$  are not isomorphic when  $i \neq j$ .

**Theorem 1.7.** *A finite perfect group  $G$  has nonisomorphic 2-proper Smith equivalent representations if and only if  $IO'(G) \neq 0$ . A finite perfect group  $G$  has a smooth action on a sphere with a given number  $k \geq 3$  of fixed points at which the tangent representations are mutually nonisomorphic if and only if  $IO'(G) \neq 0$ .*

*Proof.* The necessity in Theorem 1.7 follows from Lemma 1.4 and Remark 1.5. In order to prove the sufficiency, assume  $IO'(G) \neq 0$  and, for any integer  $k \geq 2$ , take the mutually nonisomorphic representations  $V_1, \dots, V_k$  of  $G$  determined in Example 1.6. Theorem 1.3 now asserts that for some real  $G$ -oriented representation  $W$  with  $\dim W^G = 0$ , there exists a smooth action of  $G$  on  $S^n$  with exactly  $k$  fixed points at which the tangent representations are isomorphic to  $V_1 \oplus W, \dots, V_k \oplus W$ . In particular, the tangent representations are mutually nonisomorphic. Moreover, for each element  $g \in G$  of prime power order,  $(S^n)^g$  is connected, and thus the action of  $G$  on  $S^n$  is 2-proper.  $\square$

Now, for a finite nontrivial perfect group  $G$ , we describe a subgroup of  $RO(G)$  whose elements all occur as differences of 2-proper Smith equivalent representations of  $G$ . First, for a finite group  $G$ , consider the subgroups of the complex representation ring  $R(G)$ :

$$I(G) = \bigcap_p \text{Ker}(R(G) \rightarrow R(G_p)) \quad \text{and} \quad I'(G) = I(G) \cap \text{Ker}(R(G) \xrightarrow{d^G} \mathbb{Z}).$$

Let  $r : R(G) \rightarrow RO(G)$  be the realification map. Then the following corollary holds.

**Corollary 1.8.** *Let  $G$  be a finite nontrivial perfect group. Then each element  $\alpha$  of the image  $r(I'(G))$  is a difference of 2-proper Smith equivalent representations of  $G$ .*

*Proof.* Assume  $\alpha \in r(I'(G))$ . Then  $\alpha = r(U - V)$  for two complex representations of  $G$  such that  $\dim U^G = \dim V^G$  and  $U - V \in I(G)$ . The realifications  $U_0 = r(U - U^G)$  and  $V_0 = r(V - V^G)$  are real  $G$ -oriented representations such that  $\dim U_0^G = \dim V_0^G = 0$  and  $U_0 - V_0 \in IO(G)$ . Thus, by Theorem 1.3, there exists a smooth action of  $G$  on a sphere  $S^n$  with exactly two fixed points  $x$  and  $y$ , such that as representations of  $G$ ,  $T_x(S^n) \cong U_0 \oplus W$  and  $T_y(S^n) \cong V_0 \oplus W$  for some real  $G$ -oriented representation  $W$  with  $\dim W^G = 0$ . Moreover, for each element  $g \in G$  of prime power order,  $(S^n)^g$  is connected, and thus the action of  $G$  on  $S^n$  is 2-proper. Clearly, in  $RO(G)$ ,

$$\alpha = r(U - V) = r(U - U^G) - r(V - V^G) = U_0 - V_0 = (U_0 \oplus W) - (V_0 \oplus W).$$

Therefore,  $\alpha$  is a difference of 2-proper Smith equivalent representations of  $G$ .  $\square$

## PART 2. ALGEBRA

For a finite group  $G$ , we wish to compute  $IO'(G)$ . As a subgroup of  $RO(G)$ , the group  $IO'(G)$  is free abelian and determined by its rank  $\text{rk } IO'(G)$ . Two elements  $g, h \in G$  are called *real conjugate in  $G$*  if  $h$  is conjugate to  $g$  or  $g^{-1}$ . The real conjugacy class of  $g$  is the sum  $(g)^{\pm 1} := (g) \cup (g^{-1})$  of the conjugate classes of  $g$  and  $g^{-1}$ . Recall also that  $r_G$  denotes the number of real conjugacy classes of elements of  $G$  not of prime power order.

**Lemma 2.1.** *Let  $G$  be a finite group. Then  $\text{rk } IO(G) = r_G$  and  $\text{rk } IO'(G) = r_G - 1$  when  $IO(G) \neq 0$ . Otherwise,  $IO(G) = IO'(G) = 0$ . Hence,  $IO(G) \neq 0$  if and only if  $r_G \geq 1$ ,  $IO'(G) \neq 0$  if and only if  $r_G \geq 2$ .*

*Proof.* First, we compute the rank of  $IO(G) = \bigcap_p \text{Ker}(RO(G) \rightarrow RO(G_p))$ . As the groups  $IO(G)$ ,  $RO(G)$ ,  $RO(G_p)$  are free abelian, we can tensor with  $\mathbb{R}$  to pass to the real vector spaces

$$\mathbb{R} \otimes_{\mathbb{Z}} IO(G), \quad \mathbb{R} \otimes_{\mathbb{Z}} RO(G), \quad \mathbb{R} \otimes_{\mathbb{Z}} RO(G_p),$$

to compute  $\text{rk } IO(G) = \dim_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Z}} IO(G))$ . By [Se2, Cor. 1, p. 96],  $\mathbb{R} \otimes_{\mathbb{Z}} RO(G)$  is the space of all real valued functions  $f$  on  $G$  that are constant on the real conjugacy classes  $(g)^{\pm 1}$  for  $g \in G$ . Restriction to  $\mathbb{R} \otimes_{\mathbb{Z}} RO(G_p)$  means restriction to the classes  $(g)^{\pm 1}$  with  $g$  of  $p$ -power order. Hence,  $\mathbb{R} \otimes_{\mathbb{Z}} IO(G)$  consists of the functions  $f$  which vanish on the classes  $(g)^{\pm 1}$  with  $g$  of prime power order. Thus,  $\text{rk } IO(G) = \dim_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Z}} IO(G)) = r_G$ .

Now, we compute the rank of  $IO'(G) = \text{Ker}(IO(G) \xrightarrow{d^G} \mathbb{Z})$ . A basis of  $\mathbb{R} \otimes_{\mathbb{Z}} IO(G)$  consists of the functions  $f_{(g)^{\pm 1}}$  which have the value 1 on  $(g)^{\pm 1}$  and 0 otherwise, defined for all classes  $(g)^{\pm 1}$  of elements  $g$  not of prime power order. By the fixed point set character formula (by linearity) extendable to  $\mathbb{R} \otimes_{\mathbb{Z}} RO(G)$ ,

$$d^G(f_{(g)^{\pm 1}}) = \langle f_{(g)^{\pm 1}}, 1_G \rangle = \frac{1}{|G|} \sum_{x \in G} f_{(g)^{\pm 1}}(x) = \frac{|(g)^{\pm 1}|}{|G|} \neq 0.$$

Thus,  $\text{rk } IO'(G) = r_G - 1$  when  $IO(G) \neq 0$ . Clearly,  $IO(G) = 0$  implies  $IO'(G) = 0$ . □

**Proposition 2.2.** *For  $G = A_n$ ,  $n \geq 1$ , the following two statements are true.*

- (1)  $r_G = 0$  for  $n \leq 6$ ,  $r_G = 1$  for  $n = 7$ ,  $r_G \geq 2$  for  $n \geq 8$ .
- (2) For  $n \leq 9$ ,  $G$  has no element of order 8.

*Proof.* As each element of  $A_n$  has prime power order for  $n \leq 6$ ,  $r_G = 0$  for  $n \leq 6$ . For  $n = 7$ ,  $r_G = 1$  corresponding to the element (12)(34)(567) of order 6. For  $n \geq 8$ ,  $r_G \geq 2$  because the elements (12)(34)(567) and (123456)(78), both of order 6, are not real conjugate in  $G$ , proving (1).

Any element of  $A_n$  of order 8 must involve an 8-cycle in its cycle decomposition. Thus, for  $n \leq 7$ ,  $A_n$  has no element of order 8. Also,  $A_8$  and  $A_9$  have no elements of order 8 because an 8-cycle is not an even permutation, proving (2). □

**Proposition 2.3.** *For  $G = \text{SL}_2(\mathbb{F}_p)$ ,  $p$  prime, the following two statements are true.*

- (1)  $r_G = 0$  for  $p = 2$ ,  $r_G = 1$  for  $p = 3$ ,  $r_G \geq 2$  for  $p \geq 5$ .
- (2) For  $p \leq 3$ ,  $G$  has no element of order 8.

*Proof.* As  $\text{SL}_2(\mathbb{F}_2) \cong S_3$ , the symmetric group on three letters,  $r_G = 0$  for  $G = \text{SL}_2(\mathbb{F}_2)$ . Assume  $p$  is odd and set  $G = \text{SL}_2(\mathbb{F}_p)$ . Then  $G$  has three basic cyclic subgroups of orders  $2p$ ,  $p - 1$ , and  $p + 1$  whose conjugates cover the whole group  $G$ ; see [H, S. II 8.5] for the corresponding statement for  $\text{PSL}_2(\mathbb{F}_p)$ . If  $p = 3$ , then  $p - 1 = 2$ ,  $p + 1 = 4$ , and  $2p = 6$ . The generators of a cyclic subgroup of  $G$  of order 6 are real conjugate in  $G$ . Thus,  $r_G = 1$  for  $p = 3$ . For  $p \geq 5$ , the numbers  $p - 1$  and  $p + 1$  cannot both be prime powers, because then they should be powers of 2 as they are even, which would imply  $p = 3$ . This shows that  $G$  contains an element of order  $2p$ , not a prime power, and another element not of prime power order, such that the two elements are not real conjugate in  $G$ . Thus,  $r_G \geq 2$  for  $p \geq 5$ , proving (1).

The Sylow 2-subgroup of  $G = \mathrm{SL}_2(\mathbb{F}_p)$  is isomorphic to  $\mathbb{Z}_2$  for  $p = 2$ , and the quaternion group of order 8 for  $p = 3$ . Thus, for  $p \leq 3$ ,  $G$  has no element of order 8, proving (2).  $\square$

**Proposition 2.4.** *For  $G = \mathrm{PSL}_2(\mathbb{F}_p)$ ,  $p$  prime, the following two statements are true.*

- (1)  $r_G = 0$  for  $p \leq 7$  and  $p = 17$ ,  $r_G = 1$  for  $p = 11$  and  $p = 13$ ,  $r_G \geq 2$  for  $p \geq 19$ .
- (2) For  $p \leq 13$ ,  $G$  has no element of order 8.

*Proof.* As  $\mathrm{PSL}_2(\mathbb{F}_2) = \mathrm{SL}_2(\mathbb{F}_2)$ ,  $r_G = 0$  for  $G = \mathrm{PSL}_2(\mathbb{F}_2)$ . Assume  $p$  is odd and set  $G = \mathrm{PSL}_2(\mathbb{F}_p) = \mathrm{SL}_2(\mathbb{F}_p)/\{I, -I\}$ . Similarly as in the proof of Proposition 2.3, the group  $G$  has three basic cyclic subgroups of orders  $p$ ,  $\frac{p-1}{2}$ , and  $\frac{p+1}{2}$  whose conjugates cover the whole group  $G$ . The normalizers of the subgroups of orders  $\frac{p-1}{2}$  and  $\frac{p+1}{2}$  are dihedral groups of orders  $p-1$  and  $p+1$ , respectively. This means that if we find a cyclic subgroup  $C$  of  $G$  of order not a prime power and larger than 6, then  $r_G \geq 2$  because the generators (elements not of prime power order) of  $C$  form at least two real conjugacy classes in  $G$ . For a cyclic subgroup  $C$  of  $G$  of order 6, the generators of  $C$  are real conjugate in  $G$ .

For  $p \leq 7$  and  $p = 17$ ,  $\frac{p-1}{2}$  and  $\frac{p+1}{2}$  both are prime powers, and thus  $r_G = 0$ . For  $p = 11$  and  $p = 13$ ,  $\frac{p-1}{2}$  and  $\frac{p+1}{2}$  yield a prime power and 6, and thus  $r_G = 1$ . As we check now, for  $p \geq 19$ ,  $\frac{p-1}{2}$  and  $\frac{p+1}{2}$  cannot both be prime powers, and thus  $r_G \geq 2$ . The difference of  $\frac{p+1}{2}$  and  $\frac{p-1}{2}$  is 1. If  $\frac{p-1}{2}$  and  $\frac{p+1}{2}$  both are prime powers for  $p \geq 5$ , it follows that both are greater than or equal to 2, and one is a power of 2. Then, either  $p = 2^b + 1$ ,  $b = 2^a$ , and  $p$  is a Fermat prime, or  $p = 2^a - 1$ ,  $a$  is an odd prime, and  $p$  is a Mersenne prime [HW, p. 15]. In the Fermat prime case,  $\frac{p+1}{2} = 2^{2^a-1} + 1$  is divisible by 3 as  $2^a - 1$  is odd, and thus  $\frac{p+1}{2} = 3^m$ , which implies  $3^m - 1 = 2^{2^a-1}$ . In the Mersenne prime case,  $\frac{p-1}{2} = 2^{a-1} - 1$  is divisible by 3 as  $a - 1$  is even, and thus  $\frac{p-1}{2} = 3^n$ , which implies  $3^n + 1 = 2^{a-1}$ . This is possible only for  $m = 1$  ( $p = 5$ ),  $m = 2$  ( $p = 17$ ), and  $n = 1$  ( $p = 7$ ). In fact, it follows from [Se1, Lemme, p. 32] that the order of 3 in the units of  $\mathbb{Z}_{2^k}$  divided by  $\langle -1 \rangle$  is  $2^{k-2}$  for  $k \geq 3$ . Hence, if  $2^k$  divides  $3^m - 1$  or  $3^n + 1$ , it follows that  $m, n \geq 2^{k-2}$  unless  $m = n = 1$ , and then  $3^m - 1 > 2^k$  unless  $k = 3$  and  $m = 2$ , proving (1).

The Sylow 2-subgroup of  $G = \mathrm{PSL}_2(\mathbb{F}_p)$  is isomorphic to  $\mathbb{Z}_2$  for  $p = 2$ ,  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$  for  $p = 3, 5, 11, 13$ , and the dihedral group of order 8 for  $p = 7$ . Thus, for  $p \leq 13$ ,  $G$  has no element of order 8, proving (2).  $\square$

A finite group  $G$  is called 2-*proper* if  $G$  has no element of order 8 or, for each element  $g \in G$  of order  $2^a$  with  $a \geq 3$ ,  $\dim V^g > 0$  for any irreducible representation  $V$  of  $G$ .

**Example 2.5.** Let  $G = A_n$ ,  $\mathrm{SL}_2(\mathbb{F}_p)$ , or  $\mathrm{PSL}_2(\mathbb{F}_q)$  for an integer  $n \leq 9$ , a prime  $p \leq 3$ , or a prime  $q \leq 17$ . Then  $G$  is 2-proper. In fact, except for the case  $G = \mathrm{PSL}_2(\mathbb{F}_{17})$ ,  $G$  has no element of order 8 by Propositions 2.2, 2.3, 2.4, and thus  $G$  is 2-proper. As the Sylow 2-subgroup of  $\mathrm{PSL}_2(\mathbb{F}_{17})$  is dihedral of order 16,  $\mathrm{PSL}_2(\mathbb{F}_{17})$  contains an element of order 8 but does not contain an element of order  $2^a$  with  $a > 3$ . From the atlas of finite groups, one can see that for any irreducible representation of  $\mathrm{PSL}_2(\mathbb{F}_{17})$ , each element of order 8 has positive dimensional fixed point set, and thus  $\mathrm{PSL}_2(\mathbb{F}_{17})$  is 2-proper.

**Lemma 2.6.** *Let  $G$  be a finite 2–proper group. If  $G$  acts smoothly on a homotopy sphere  $\Sigma$  with  $\Sigma^G \neq \emptyset$ , then the action is 2–proper. In particular, any Smith equivalent representations of  $G$  are 2–proper Smith equivalent.*

*Proof.* If  $G$  has no element of order 8 and hence no element of order  $2^a$  with  $a \geq 3$ , then the action is 2–proper. Let  $g$  be an element of  $G$  of order  $2^a$  with  $a \geq 3$ . Choose  $x \in \Sigma^G$  and set  $U = T_x(\Sigma)$ . By Smith theory, either  $\Sigma^g = \{x, y\}$  for a point  $y \in \Sigma$ , or  $\Sigma^g$  is connected. By the Slice Theorem,  $\dim \Sigma^g = \dim U^g \geq \dim V^g > 0$  for any irreducible summand  $V$  of  $U$ . Thus,  $\Sigma^g$  is connected and the action is 2–proper.  $\square$

**Theorem 2.7.** *Let  $G = A_n, \mathrm{SL}_2(\mathbb{F}_p)$ , or  $\mathrm{PSL}_2(\mathbb{F}_q)$  for an integer  $n \geq 1$  and primes  $p, q$ . If  $r_G \leq 1$ , then for any smooth action of  $G$  on a homotopy sphere, at any two isolated fixed points of the action, the tangent representations of  $G$  are isomorphic. If  $r_G \geq 2$ , then there exist nonisomorphic 2–proper Smith equivalent representations of  $G$ . If  $r_G \geq 2$ , then there exists a smooth action of  $G$  on a standard sphere with any given number  $k \geq 3$  of fixed points at which the tangent representations of  $G$  are mutually nonisomorphic.*

*Proof.* Assume  $r_G \leq 1$ , which by Lemma 2.1 means that  $IO'(G) = 0$ . It follows from Propositions 2.2, 2.3, 2.4 that  $n \leq 7, p \leq 3, q \leq 17$ , and by Example 2.5,  $G$  is 2–proper. For a smooth action of  $G$  on a homotopy sphere  $\Sigma$  with isolated points  $x$  and  $y$  of  $\Sigma^G$ , set  $U = T_x(\Sigma)$  and  $V = T_y(\Sigma)$ . By Lemma 2.6, the action of  $G$  on  $\Sigma$  is 2–proper, and by Lemma 1.4 and Remark 1.5,  $U - V \in IO'(G) = 0$ ; that is,  $U$  and  $V$  are isomorphic.

Assume  $r_G \geq 2$ , which by Lemma 2.1 means that  $IO'(G) \neq 0$ . Now, it follows from Propositions 2.2, 2.3, 2.4 that  $n \geq 8, p \geq 5, q \geq 19$ . Thus,  $G$  is perfect, and Theorem 1.7 completes the proof.  $\square$

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#### APPENDIX (ADDED BY KRZYSZTOF PAWAŁOWSKI)

On Saturday, August 24, 1996, Erkki Laitinen died suddenly. We were in the middle of work on the question of Smith equivalence of representations. Just a few days before his death, Erkki Laitinen posed the following conjecture for a finite Oliver group; that is, a finite group  $G$  not of prime power order with  $n_G = 1$ .

**Laitinen Conjecture.**  *$G$  has nonisomorphic 2–proper Smith equivalent representations if and only if  $r_G \geq 2$ .*

By Theorem A, the Laitinen Conjecture holds for any finite perfect group  $G$ , and by Theorem B, the same is true for the groups  $A_n, \mathrm{SL}_2(\mathbb{F}_p), \mathrm{PSL}_2(\mathbb{F}_q)$  (recall

that these groups are not perfect for  $n = 3, 4$  and primes  $p, q < 5$ ). The condition that  $r_G \geq 2$  is necessary in the Laitinen Conjecture, because Lemmas 1.4, 2.1, and 2.6 yield the following lemma.

**Laitinen Lemma.** *For a finite group  $G$  with  $r_G \leq 1$ , any 2-proper Smith equivalent representations of  $G$  are isomorphic. Moreover, for a finite 2-proper group  $G$  with  $r_G \leq 1$ , any Smith equivalent representations of  $G$  are isomorphic.*

*Remark.* [Sch, Comment (2), p. 547] contains the speculation: it is likely that finite Oliver groups  $G$  all have nonisomorphic Smith equivalent representations. As follows from our results, this is not true. In fact,  $A_n$  for  $n = 5, 6, 7$ , and  $\mathrm{PSL}_2(\mathbb{F}_q)$  for  $q = 5, 7, 11, 13, 17$ , all are perfect (Oliver) 2-proper groups  $G$  with  $r_G \leq 1$ , and the Laitinen Lemma asserts that any Smith equivalent representations of  $G$  are isomorphic.

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