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ON PRINCIPAL EIGENVALUES FOR BOUNDARY VALUE PROBLEMS WITH INDEFINITE WEIGHT AND ROBIN BOUNDARY CONDITIONS

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Abstract. We investigate the existence of principal eigenvalues (i.e., eigenvalues corresponding to positive eigenfunctions) for the boundary value problem $-\Delta u(x) = \lambda g(x)u(x)$ on D; $\frac{\partial u}{\partial n}(x) + \alpha u(x) = 0$ on ∂D, where D is a bounded region in $\mathbf{R}^N$, g is an indefinite weight function and $\alpha \in \mathbf{R}$ may be positive, negative or zero.

We discuss the existence of principal eigenvalues (i.e., eigenvalues corresponding to positive eigenfunctions) for the boundary value problem

$$(1)_\alpha \qquad -\Delta u(x) = \lambda g(x)u(x) \text{ on } D; \qquad \frac{\partial u}{\partial n}(x) + \alpha u(x) = 0 \text{ on } \partial D,$$

where D is a bounded region in $\mathbf{R}^N$ with smooth boundary, $g : D \to \mathbf{R}$ is a smooth function which changes sign on D, and $\alpha \in \mathbf{R}$.

Such problems have been studied in recent years because of associated nonlinear problems arising in the study of population genetics (see [3]). The study of the linear ordinary differential equation case, however, goes back to Picone and Bôcher (see [2]). Attention has been confined mainly to the cases of Dirichlet ($\alpha = \infty$) and Neumann boundary conditions.

In the case of Dirichlet boundary conditions it is well known (see [4]) that there exists a double sequence of eigenvalues for $(1)_\alpha$

$$\dots \lambda_2^- < \lambda_1^- < 0 < \lambda_1^+ < \lambda_2^+ \dots,$$

λ_1^+ (λ_1^-) being the unique positive (negative) principal eigenvalue. It is also well known that the case where $0 < \alpha < \infty$ is similar to the Dirichlet case. In the case of Neumann boundary conditions, 0 is clearly a principal eigenvalue and there is a positive (negative) principal eigenvalue if and only if $\int_D g(x)\,dx < 0\,(> 0)$; in the case where $\int_D g(x) = 0$ there are no positive and no negative principal eigenvalues.

We shall investigate how the principal eigenvalues of $(1)_\alpha$ depend on α, obtaining new results for the case where $\alpha < 0$. This case seems to have been considered far less often than the case $\alpha \geq 0$, probably because it is more natural that the flux across the boundary should be outwards if there is a positive concentration at the

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boundary, and also because $\alpha \geq 0$ is an easier condition to use when applying the maximum principle to discuss positive solutions. By studying the case $\alpha < 0$, however, we obtain a much clearer overall view of how the principal eigenvalues of $(1)_\alpha$ depend on α. We shall show that, depending on α, $(1)_\alpha$ has two, one or zero principal eigenvalues, and that the natural way of distinguishing between principal eigenvalues is by considering the sign of $\int_D g(x)u_0^2\,dx$, where u_0 denotes the corresponding eigenfunction rather than the sign of the eigenvalues themselves.

Our analysis is based on a method used by Hess and Kato ([4]). Consider, for fixed λ, the eigenvalue problem

$$(2)_\alpha \qquad -\Delta u(x) - \lambda g(x)u(x) = \mu u(x) \text{ on } D; \qquad \frac{\partial u}{\partial n}(x) + \alpha u(x) = 0 \;\text{ on } \partial D.$$

We denote the lowest eigenvalue of $(2)_\alpha$ by $\mu(\alpha,\lambda)$. Let

$$S_{\alpha,\lambda} = \{\int_D |\nabla\phi|^2\,dx + \alpha\int_{\partial D}\phi^2\,dS_x - \lambda\int_D g\phi^2\,dx \;:\; \phi \in W^{1,2}(D)\,,\; \int_D \phi^2\,dx = 1\}.$$

When $\alpha \geq 0$, it is clear that $S_{\alpha,\lambda}$ is bounded below. It is shown in Smoller [5] by variational arguments that $\mu(\alpha,\lambda) = \inf S_{\alpha,\lambda}$ and that an eigenfunction corresponding to $\mu(\alpha,\lambda)$ does not change sign on D. Thus, clearly, λ is a principal eigenvalue of $(1)_\alpha$ if and only if $\mu(\alpha,\lambda) = 0$.

When $\alpha < 0$, the boundedness below of $S_{\alpha,\lambda}$ is no longer obvious *a priori*, but is a consequence of the following lemma.

Lemma 1. *For every $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that*

$$\int_{\partial D}\phi^2\,dS_x \leq \epsilon\int_D |\nabla\phi|^2\,dx + C(\epsilon)\int_D \phi^2\,dx$$

for all $\phi \in W^{1,2}(D)$.

Proof. Suppose that the result does not hold. Then there exist $\epsilon_0 > 0$ and a sequence $\{u_n\} \subseteq W^{1,2}(D)$ such that $\int_D |\nabla u_n|^2\,dx = 1$ and

$$(3) \qquad \int_{\partial D} u_n^2\,dS_x \geq \epsilon_0 + n\int_D u_n^2\,dx.$$

Suppose first that $\{\int_D u_n^2\,dx\}$ is unbounded. Let $v_n = u_n/||u_n||_{L_2(D)}$. Clearly $\{v_n\}$ is bounded in $W^{1,2}(D)$, and so in $L^2(\partial D)$. But $\int_{\partial D} v_n^2\,dS_x \geq n\int_D v_n^2\,dx = n$, which is impossible.

Suppose now that $\{\int_D u_n^2\,dx\}$ is bounded. Then $\{u_n\}$ is bounded in $W^{1,2}(D)$ and so has a subsequence, which we again denote by $\{u_n\}$, converging weakly to u in $W^{1,2}(D)$. Since $W^{1,2}(D)$ is compactly embedded in $L^2(\partial D)$ (see Adams [1], page 144) and in $L^2(D)$, it follows that $\{u_n\}$ converges to some function u in $L^2(\partial D)$ and in $L^2(D)$. Thus $\{\int_{\partial D} u_n^2\,dS_x\}$ is bounded, and so it follows from (3) that $\lim_{n\to\infty}\int_D u_n^2\,dx = 0$, i.e., $\{u_n\}$ converges to zero in $L^2(D)$. Hence $\{u_n\}$ converges to 0 in $L^2(\partial D)$, and this is impossible because of (3). □

Choosing $\epsilon < \frac{1}{\alpha}$, it is easy to deduce from the above result that $S_{\alpha,\lambda}$ is bounded below, and it follows exactly as in [5] that $\mu(\alpha,\lambda) = \inf S_{\alpha,\lambda}$ and that an eigenfunction corresponding to $\mu(\alpha,\lambda)$ does not change sign on D. Thus it is again the case that λ is a principal eigenvalue of $(1)_\alpha$ if and only if $\mu(\alpha,\lambda) = 0$.

For fixed $\phi \in W^{1,2}(D)$, $\lambda \to \int_D |\nabla\phi|^2\,dx + \alpha\int_{\partial D}\phi^2\,dS_x - \lambda\int_D g\phi^2\,dx$ is an affine and so concave function. As the infimum of any collection of concave functions is concave, it follows that $\lambda \to \mu(\alpha,\lambda)$ is a concave function. Also, by considering test

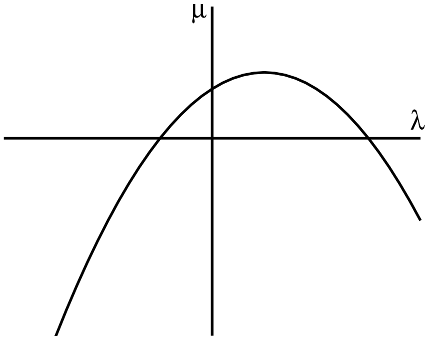

FIGURE 1. Graph of $\lambda \to \mu(\alpha, \lambda)$ when $\alpha > 0$.

functions $\phi_1, \phi_2 \in W^{1,2}(D)$ such that $\int_D g\phi_1^2\,dx > 0$ and $\int_D g\phi_2^2\,dx < 0$, it is easy to see that $\mu(\alpha,\lambda) \to -\infty$ as $\lambda \to \pm\infty$. Thus $\lambda \to \mu(\alpha,\lambda)$ is an increasing function until it attains its maximum, and is a decreasing function thereafter.

Suppose that $0 < \alpha < \infty$, i.e., we have the 'usual' Robin boundary condition. Then, as can be seen from the variational characterisation of $\mu(\alpha,\lambda)$ or the fact that $-\Delta$ has a positive principal eigenvalue, $\mu(\alpha, 0) > 0$ and so $\lambda \to \mu(\alpha,\lambda)$ must have a graph similar to that shown in Figure 1, i.e., $\lambda \to \mu(\alpha,\lambda)$ has exactly two zeros. Thus in this case $(1)_\alpha$ has exactly two principal eigenvalues, one positive and one negative.

In the case $\alpha \leq 0$ we have that $\mu(\alpha, 0) \leq 0$, and the situation is less clear.

Lemma 2. *Suppose that* u_0 *is an eigenfunction of* $(2)_\alpha$ *corresponding to the principal eigenvalue* $\mu(\alpha,\lambda)$. *Then*

$$\frac{d\mu}{d\lambda}(\alpha,\lambda) = -\frac{\int_D g u_0^2\,dx}{\int_D u_0^2\,dx}.$$

Proof. Regarding u and μ as functions of λ, we have

$$-\Delta u(\lambda) - \lambda g(x)u(\lambda) = \mu(\lambda)u(\lambda) \text{ on } D; \qquad \frac{\partial}{\partial n}u(\lambda) + \alpha u(\lambda) = 0 \text{ on } \partial D.$$

Let $v(\lambda) = \frac{du}{d\lambda}$. Then $v(\lambda)$ satisfies

$$-\Delta v(\lambda) - \lambda g(x)v(\lambda) - \mu(\lambda)v(\lambda) = g(x)u(\lambda) + \frac{d\mu}{d\lambda}(\lambda)u(\lambda) \text{ on } D. \tag{4}$$

In addition, $\frac{\partial}{\partial n}v(\lambda) + \alpha v(\lambda) = 0$ on ∂D.

Multiplying (4) by $u(\lambda)$ and integrating over D gives

$$0 = \int_D g(x)[u(\lambda)]^2\,dx + \frac{d\mu}{d\lambda}(\lambda)\int_D [u(\lambda)]^2\,dx$$

and so the result follows. □

The above lemma shows that where $\lambda \to \mu(\alpha,\lambda)$ is an increasing (decreasing) function we have that $\int_D g(x)[u(\lambda)]^2\,dx < 0\,(> 0)$, and at critical points we must have $\int_D g(x)[u(\lambda)]^2\,dx = 0$. The next lemma shows that $\lambda \to \mu(\alpha,\lambda)$ has a unique critical point.

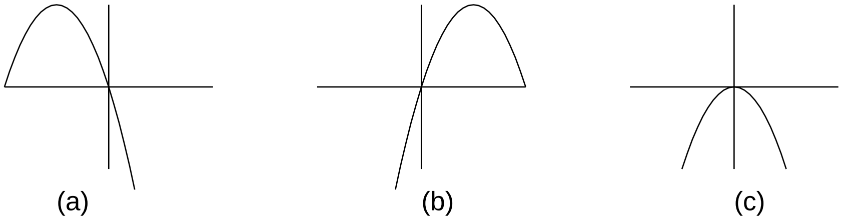

FIGURE 2. Graph of $\lambda \to \mu(\alpha, \lambda)$ when $\alpha = 0$ in the cases where (a) $\int_D g > 0$, (b) $\int_D g < 0$, (c) $\int_D g = 0$.

Lemma 3. *Suppose that u_0 is an eigenfunction of $(2)_\alpha$ corresponding to the principal eigenvalue $\mu(\alpha, \lambda_0)$ such that $\int_D g(x) u_0^2 \, dx = 0$. Then $\mu(\alpha, \lambda_0) > \mu(\alpha, \lambda)$ whenever $\lambda \neq \lambda_0$, i.e., the unique global maximum of $\lambda \to \mu(\alpha, \lambda_0)$ occurs when $\lambda = \lambda_0$.*

Proof. We may assume without loss of generality that $\int_D u_0^2 \, dx = 1$. Then

$$\mu(\alpha, \lambda_0) = \int_D |\nabla u_0|^2 \, dx + \alpha \int_{\partial D} u_0^2 \, dS_x.$$

Hence

$$\begin{aligned} \mu(\alpha, \lambda) &\leq \int_D |\nabla u_0|^2 \, dx + \alpha \int_{\partial D} u_0^2 \, dS_x - \lambda \int_D g(x) u_0^2 \, dx \\ &= \int_D |\nabla u_0|^2 \, dx + \alpha \int_D u_0^2 \, dS_x = \mu(\alpha, \lambda_0). \end{aligned}$$

Suppose $\lambda \neq \lambda_0$ and $\mu(\alpha, \lambda) = \mu(\alpha, \lambda_0)$. Then u_0 is a minimizer for $S_{\alpha,\lambda}$, and it follows that u_0 must satisfy

$$-\Delta u_0(x) - \lambda g(x) u_0(x) = \mu(\alpha, \lambda) u_0(x) \text{ on } D; \quad \frac{\partial u_0}{\partial n}(x) + \alpha u_0(x) = 0 \text{ on } \partial D.$$

But as $\mu(\alpha, \lambda) = \mu(\alpha, \lambda_0)$, u_0 also satisfies

$$-\Delta u_0(x) - \lambda_0 g(x) u_0(x) = \mu(\alpha, \lambda) u_0(x) \text{ on } D; \quad \frac{\partial u_0}{\partial n}(x) + \alpha u_0(x) = 0 \text{ on } \partial D,$$

and this is a contradiction. Hence $\mu(\alpha, \lambda) < \mu(\alpha, \lambda_0)$. ☐

Thus $\lambda \to \mu(\alpha, \lambda)$ is a concave function which is increasing on some interval of the form $(-\infty, \hat{\lambda})$, has a maximum turning point at $\lambda = \hat{\lambda}$, and is decreasing on $(\hat{\lambda}, \infty)$. Hence the graph of $\lambda \to \mu(\alpha, \lambda)$ may have $2, 1$ or 0 intersections with the μ-axis, and so $(1)_\alpha$ may have $2, 1$ or 0 principal eigenvalues.

We have already seen that when $\alpha > 0$, $(1)_\alpha$ has 2 principal eigenvalues, one positive and one negative. If $\alpha = 0$, i.e., we have Neumann boundary conditions, then $\mu(\alpha, 0) = 0$ and the corresponding eigenfunction is a constant. Hence $\frac{d\mu}{d\lambda}(0) > 0 \, (= 0) \, (< 0)$ as $\int_D g(x) \, dx < 0 \, (= 0) \, (> 0)$. Thus, when $\alpha = 0$, $\mu = 0$ is a principal eigenvalue in all cases; if $\int_D g(x) \, dx < 0$, there is an additional positive principal eigenvalue; and, if $\int_D g(x) \, dx > 0$, there is an additional negative principal eigenvalue and, if $\int_D g(x) \, dx = 0$, $\mu = 0$ is the only principal eigenvalue (see Figure 2).

We now consider what happens when $\alpha < 0$. We first assume that $\int_D g(x) \, dx < 0$. It is clear from the variational characterisation of $\mu(\alpha, \lambda)$ that $\alpha \to \mu(\alpha, \lambda)$ is a

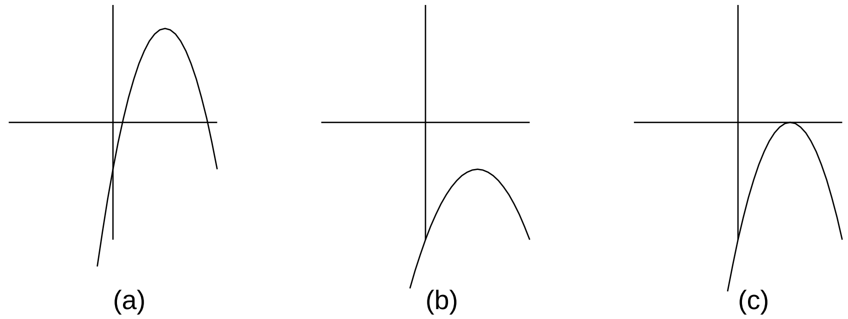

FIGURE 3. Graph of $\lambda \to \mu(\alpha, \lambda)$ when $\int g < 0$ and $\alpha < 0$, where (a) α is small, (b) α is large, and (c) $\alpha = \alpha_0$.

strictly increasing, concave (and so continuous) function. Thus, for α sufficiently small and negative, $\lambda \to \mu(\alpha, \lambda)$ must have a graph of the form shown in Figure 3(a), and so $(1)_\alpha$ has two positive principal eigenvalues. This state of affairs does not persist, however, for all $\alpha < 0$.

Lemma 4. *There exists $\alpha^* < 0$ such that $(1)_\alpha$ has no principal eigenvalues if $\alpha < \alpha^*$.*

Proof. Suppose $\alpha < 0$ and u_0 is a positive eigenfunction of $(1)_\alpha$ corresponding to a positive principal eigenvalue λ_0. It is easy to show by using the maximum principle that $u_0(x) > 0$ for all $x \in \overline{D}$. Also $0 = \mu(\alpha, \lambda_0) < \mu(0, \lambda_0)$. Hence $\lambda_0 < \mu_0$ (the positive principal eigenvalue for the Neumann problem.)

Dividing $(1)_\alpha$ by u_0 and integrating over D, we have

$$\int_D \frac{-\Delta u_0}{u_0}\, dx = \lambda_0 \int_D g(x)\, dx,$$

and so

$$-\int_{\partial D} \frac{\partial u_0}{\partial n}\, u_0\, dS_x - \int_D \frac{|\nabla u_0|^2}{u_0^2}\, dx = \lambda_0 \int_D g(x)\, dx,$$

i.e.,

$$\alpha \int_{\partial D} dS_x - \int_D \frac{|\nabla u_0|^2}{u_0^2}\, dx = \lambda_0 \int_D g(x)\, dx.$$

Hence $\alpha = (\lambda_0 \int_D g(x)\, dx + \int_D \frac{|\nabla u_0|^2}{u_0^2}\, dx) \,/\, |\partial D|$. Since $\lambda_0 < \mu_0$, α cannot be too negative, and the proof is complete. □

It follows that for large negative α the graph of $\lambda \to \mu(\alpha, \lambda)$ must be as in Figure 3(b), and so by the continuity of $\alpha \to \mu(\alpha, \lambda)$ there must exist α_0 such that $\max_\lambda \mu(\alpha_0, \lambda) = 0$ (see Figure 3(c)). Clearly $(1)_{\alpha_0}$ has precisely one principal eigenvalue.

A similar analysis can be carried out in the case $\int_D g(x)\, dx > 0$; in this case two negative principal eigenvalues will occur for an appropriate range of negative α.

Our results may be summarized in the following theorem.

Theorem 5. *There exists $\alpha_0 \le 0$ such that*

(*i*) *if $\alpha < \alpha_0$, then $(1)_\alpha$ does not have a principal eigenvalue;*

(ii) *if* $\alpha = \alpha_0$*, then* $(1)_\alpha$ *has a unique principal eigenvalue with corresponding eigenfunction* u_0 *such that* $\int_D g(x)u_0^2\,dx = 0$*;*

(iii) *if* $\alpha > \alpha_0$*, then* $(1)_\alpha$ *has exactly two principal eigenvalues* λ *and* μ*,* $\lambda < \mu$*; if* u_0 *and* v_0 *are eigenfunctions corresponding to* λ *and* μ*, respectively, then* $\int_D g(x)u_0^2\,dx < 0$ *and* $\int_D g(x)v_0^2\,dx > 0$*;*

(iv) $\alpha_0 = 0$ *if and only if* $\int_D g(x)\,dx = 0$*.*

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