PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 1, January 1999, Pages 167–172
S 0002-9939(99)04748-6

# REMARKS ON BLOW-UP BEHAVIOR FOR A NONLINEAR DIFFUSION EQUATION WITH NEUMANN BOUNDARY CONDITIONS

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(Communicated by Jeffrey B. Rauch)

ABSTRACT. We establish the blow-up rate for the solution of a nonlinear diffusion equation $(u^m)_t = u_{xx},\ 0 < x < 1,\ t > 0$, subject to Neumann boundary conditions $u_x(0,t) = 0,\ u_x(1,t) = u^\alpha(1,t)$.

## 1. INTRODUCTION

In this paper, we consider the initial–boundary value problem

$$(1.1)\qquad \begin{aligned} &(u^m)_t = u_{xx}, && 0 < x < 1, \quad t > 0,\\ &u_x(0,t) = 0,\ u_x(1,t) = u^\alpha(1,t), && t > 0,\\ &u(x,0) = u_0(x) \geq \delta > 0, && 0 \leq x \leq 1. \end{aligned}$$

Here $0 < m,\ \alpha < \infty$, $u_0 \in C^{2+\nu}([0,1])$ for some $0 < \nu < 1$, and $u_0'(0) = 0$, $u_0'(1) = u_0^\alpha(1)$.

Such a problem was previously studied in [5], wherein the following results were established.

*Case* 1. $0 < m < 1$. If $\alpha \leq m$, every solution of problem (1.1) exists globally while the solution blows up in a finite time $T$ for $\alpha > m$. Moreover, if $m < \alpha \leq 1$, blow-up occurs on the whole interval $[0,1]$, and there exist two positive constants $C_1$ and $C_2$ such that

$$(1.2)\qquad C_1(T-t)^{1/(m-\alpha)} \leq u(x,t) \leq C_2(T-t)^{1/(m-\alpha)} \qquad \text{for } (x,t) \in [0,1] \times [0,T).$$

If $m < 1 < \alpha$, blow-up occurs only at $x = 1$.

*Case* 2. $m > 1$. If $2\alpha < m+1$, problem (1.1) possesses a global solution, whereas for $2\alpha > m+1$, every solution must blow up in a finite time $T$. Furthermore, if $\alpha > m$, blow-up occurs only at $x = 1$. And for $2\alpha > m+1$, under an additional condition, the solution $u$ satisfies

$$(1.3)\qquad C_\epsilon(T-t)^{1/(m+1-2\alpha-\epsilon)} \leq u(1,t) \leq C(T-t)^{1/(m+1-2\alpha)} \qquad \text{for } t \in [0,T)$$

with $\epsilon \in (0,1)$ and $C_\epsilon,\ C > 0$. The condition is as follows.

Received by the editors May 6, 1997.
1991 *Mathematics Subject Classification.* Primary 35B40, 35K20, 35K55.

(**H**) For some $\lambda > 0$, $u_0(x) \in C^\infty([0,1])$ is a positive solution of the problem

$$f''(x) = \lambda m f^{m+\alpha-1}(x) \qquad 0 < x < 1,$$
$$f'(0) = 0,\ f'(1) = f^\alpha(1).$$

Later, it was shown in [8] that the critical case $2\alpha = m+1$ belongs to the global existence part. When $m = 1$, the nonlinear diffusion equation reduces to the heat equation, and (1.1) has been extensively studied (see [4], [6] and the literature cited therein).

Several questions remain open: What is the blow-up rate for the case $m < 1 < \alpha$? Compared to the result for $m = 1$ in [4], can the lower bound in (1.3) be improved? And where does blow-up occur for the case $(m+1)/2 < \alpha \le m$?

Our objective in this paper is to answer all the above questions. For convenience, letting $a(u) = (1/m)u^{1-m}$ and $g(u) = u^\alpha$, in the sequel we shall mainly concentrate on the equivalent problem

$$\begin{aligned} &u_t = a(u)u_{xx}, && 0 < x < 1, \quad 0 < t < T,\\ &u_x(0,t) = 0,\ u_x(1,t) = g(u), && 0 < t < T,\\ &u(x,0) = u_0(x), && 0 \le x \le 1. \end{aligned} \tag{1.4}$$

## 2. Blow-up rate for the case $m < 1 < \alpha$

In this section we establish bounds on the blow-up rate for $m < 1 < \alpha$. We first present the lower bound.

**Theorem 2.1.** *Let $m < 1 < \alpha$ and $u_0'(x) \ge 0$ on $[0,1]$. Then there exists a positive constant $C_3$ such that*

$$u(1,t) \ge C_3(T-t)^{1/(m+1-2\alpha)} \qquad \text{for } t \in [0,T). \tag{2.1}$$

*Proof.* Since $u_0'(x) \ge 0$ for $0 \le x \le 1$, the maximum principle yields that $u_x(x,t) \ge 0$ on $[0,1] \times [0,T)$. Inspired by [1], [3], we define a function $\Phi(x,t) = u_x(x,t) - \varphi(x)g(u(x,t))$ in $(0,1) \times (0,T)$, where $\varphi(x)$ is a nonnegative function to be determined. Through a routine calculation, we obtain

$$\begin{aligned} \Phi_t &= a(u)\Phi_{xx} + a'(u)u_x\Phi_x \\ &\quad + \varphi(x)(a'(u)g'(u) + a(u)g''(u))u_x^2 \\ &\quad + \varphi'(x)(a'(u)g(u) + 2a(u)g'(u))u_x + \varphi''(x)a(u)g(u). \end{aligned} \tag{2.2}$$

Since $a'(u)g'(u)+a(u)g''(u) = (\alpha/m)(\alpha-m)u^{\alpha-m-1} > 0$ and $a'(u)g(u)+2a(u)g'(u) = ((2\alpha-m+1)/m)u^{\alpha-m} > 0$, if $\varphi(x)$ satisfies $\varphi' \ge 0$ and $\varphi'' \ge 0$ on $(0,1)$, then $\Phi$ cannot attain its negative minimum in $(0,1) \times (0,T)$.

Because $u_0'(1) = u_0^\alpha(1)$, $u_0'(x) > 0$ near $x = 1$, we can construct $\varphi(x)$ as follows: $\varphi(x) \equiv 0$ for $0 \le x \le x_0$ with some $x_0 < 1$ and $\varphi(x) = (x-x_0)^l/(1-x_0)^l$ on $(x_0,1]$, where $l$ $(\ge 3)$ is chosen so large that $\varphi(x) \le u_0'(x)u_0^{-\alpha}(x)$ for $x_0 < x \le 1$. Then $\Phi(0,t) = \Phi(1,t) = 0$, and $\Phi(x,0) \ge 0$. By the maximum principle, we have $\Phi(x,t) \ge 0$, that is,

$$u_x(x,t) \ge \varphi(x)g(u(x,t)) \qquad \text{for } (x,t) \in [0,1] \times [0,T). \tag{2.3}$$

Moreover, $\Phi_x(1,t) \le 0$, which means

$$\begin{aligned} u_t(1,t) &= a(u(1,t))u_{xx}(1,t) \\ &\le a(u(1,t))g(u(1,t))(\varphi'(1) + g'(u(1,t))) \le c_0 u^{2\alpha-m}(t). \end{aligned} \tag{2.4}$$

Integrating (2.4) from $t$ to $T$ yields the asserted result.

We then give the upper bound, the derivation of which is in the spirit of [2].

**Theorem 2.2.** *Let $m < 1 < \alpha$ and $u_0'(x)$, $u_0''(x) \geq 0$ on $[0,1]$. Then there exists a positive constant $C_4$ such that*

$$u(1,t) \leq C_4(T-t)^{1/(m+1-2\alpha)} \qquad \text{for } t \in [0,T). \tag{2.5}$$

*Proof.* Since $u_0''(x) \geq 0$ for $0 \leq x \leq 1$, by the maximum principle one can see that $u_{xx}(x,t)$ and $u_t(x,t)$ are positive on $[0,1] \times [\tau, T)$ for any $\tau \in (0,T)$. Letting $\tau$ be close to $T$, consider $\Psi(x,t) = u_t(x,t) - \epsilon d(u(x,t))u_x^{2-\sigma}(x,t)$ in $(1-T+\tau, 1) \times(\tau, T)$, where $\epsilon$ is a positive constant, $(m+1)/\alpha < \sigma < (m+\alpha)/\alpha$, and $d(u) = a(u)g'(u)g^{\sigma-1}(u)$. By a tedious computation we find that

$$\Psi_t = a(u)\Psi_{xx} + C(x,t)\Psi + \epsilon R(x,t)u_x^{4-\sigma},$$

where

$$\begin{aligned} C(x,t) &= \left[\frac{a'(u)}{a(u)} + \epsilon(2-\sigma)(1-\sigma)\frac{d(u)}{a(u)}u_x^{-\sigma}\right]u_t \\ &\quad + \epsilon\left[(4-2\sigma)d'(u) - (1-\sigma)\frac{a'(u)d(u)}{a(u)}\right]u_x^{2-\sigma} \\ &\quad + \epsilon^2(2-\sigma)(1-\sigma)\frac{d^2(u)}{a(u)}u_x^{2-2\sigma} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} R(x,t) &= a(u)d''(u) + \epsilon\left[(\sigma-1)a'(u)d(u) + 2(2-\sigma)a(u)d'(u)\right]\frac{d(u)}{a(u)}u_x^{-\sigma} \\ &\quad + \epsilon^2(2-\sigma)(1-\sigma)\frac{d^3(u)}{a(u)}u_x^{-2\sigma}. \end{aligned} \tag{2.7}$$

Since $a(u) = (1/m)u^{1-m}$ and $d(u) = (\alpha/m)u^{\alpha\sigma-m}$, we have

$$\begin{aligned} R(x,t) &= \frac{\alpha}{m^2}\Bigg\{(\alpha\sigma-m)(\alpha\sigma-m-1) + \epsilon\alpha\left[(\sigma-1)(1-m)\right. \\ &\quad \left. + 2(2-\sigma)(\alpha\sigma-m)\right]\frac{g^\sigma(u)}{u_x^\sigma} + \epsilon^2\alpha^2(2-\sigma)(1-\sigma)\frac{g^{2\sigma}(u)}{u_x^{2\sigma}}\Bigg\}u^{\alpha\sigma-2m-1}. \end{aligned} \tag{2.8}$$

Recalling (2.3) we observe that if $1-T+\tau > x_0$,

$$u_x(x,t) \geq kg(u(x,t)) \qquad \text{for } (x,t) \in [1-T+\tau, 1] \times [\tau, T) \tag{2.9}$$

with $k$ a positive constant. Consequently, $R(x,t) \geq 0$ for small $\epsilon$, which implies

$$\Psi_t \geq a(u)\Psi_{xx} + C(x,t)\Psi \qquad \text{in } (1-T+\tau, 1) \times (\tau, T). \tag{2.10}$$

On the parabolic boundary, since $x = 1$ is the only blow-up point, if $\epsilon$ is small enough, then both $\Psi(1-T+\tau, t)$ and $\Psi(x,\tau)$ are nonnegative. At $x = 1$,

$$\Psi_x(1,t) = b(1,t)\Psi(1,t) + \epsilon g^{2-\sigma}(u(1,t))Q(u(1,t)), \tag{2.11}$$

where

$$b(x,t) = g'(u) - \epsilon(2-\sigma)\frac{d(u)}{a(u)}g^{1-\sigma}(u)$$

and

$$
\begin{aligned}
Q(u) &= d(u)g'(u) - d'(u)g(u) - \epsilon(2-\sigma)\frac{d^2(u)}{a(u)}g^{1-\sigma}(u) \\
&= \frac{\alpha}{m}(\alpha + m - \alpha\sigma - \epsilon\alpha(2-\sigma))u^{\alpha\sigma+\alpha-m-1}. \qquad (2.12)
\end{aligned}
$$

Since $\sigma < (m+\alpha)/\alpha$, if $\epsilon$ is sufficiently small, then $Q(u(1,t)) > 0$, which means that $\Psi_x(1,t) > b(1,t)\Psi(1,t)$ for $\tau \le t < T$. Thus $\Psi(x,t) \ge 0$ on $[1-T+\tau, 1] \times [\tau, T)$. In particular, $\Psi(1,t) \ge 0$, that is ,

$$
u_t(1,t) \ge \epsilon a(u(1,t))g(u(1,t))g'(u(1,t)) = \frac{\epsilon\alpha}{m}u^{2\alpha-m}(1,t). \qquad (2.13)
$$

Integration of (2.13) over $(t,T)$ then leads to (2.5).

## 3. Blow-up rate and set for the case $1 < m < 2\alpha - 1$

We begin this section by presenting the lower bound on the blow-up rate for $2\alpha > m+1$. Here we use a modification of an argument from [5].

**Theorem 3.1.** *Let $1 < m < 2\alpha - 1$ and $u_0'(x)$, $u_0''(x) \ge 0$ on $[0,1]$. Then there exists a positive constant $C_5$ such that*

$$
u(1,t) \ge C_5(T-t)^{1/(m+1-2\alpha)} \qquad \textit{for } t \in [0,T). \qquad (3.1)
$$

*Proof.* It suffices to prove (3.1) for $t \in [\eta, T)$ with some $\eta$ such that $u(1,\eta) \ge 1$. Set

$$
y(t) = u^{\gamma}(1,t)\int_{1-\xi(t)}^{1} u^m(x,t)dx \qquad (3.2)
$$

with

$$
\xi(t) = (1-\epsilon)u^{1-\alpha}(1,t), \qquad (3.3)
$$

where $0 < \epsilon < 1$, $(1-\epsilon)/\epsilon^m < 2/(\alpha-1)$, and $\alpha - m - 1 < \gamma \le (\alpha-1)[1-m(1-\epsilon)/2\epsilon^m]$. A straightforward computation gives

$$
\begin{aligned}
y'(t) &= \gamma u^{\gamma-1}(1,t)u_t(1,t)\int_{1-\xi(t)}^{1} u^m(x,t)dx \\
&\quad + u^{\gamma}(1,t)(u_x(1,t) - u_x(1-\xi(t),t)) \\
&\quad + (1-\epsilon)(1-\alpha)u^{\gamma-\alpha}(1,t)u^m(1-\xi(t),t)u_t(1,t) \qquad (3.4) \\
&\le -u^{\gamma-1}(1,t)u_t(1,t)I(t) + u^{\gamma+\alpha}(1,t),
\end{aligned}
$$

where $I(t) = (\alpha-1)u^m(1-\xi(t),t)\xi(t) - \gamma\int_{1-\xi(t)}^{1} u^m(x,t)dx$. Since $u_x \ge 0$ and $u_{xx} \ge 0$ in $[0,1] \times [\eta, T)$, we find

$$
\epsilon u(1,t) \le u(1-\xi(t),t) \le u(x,t) \le u(1,t) \qquad (3.5)
$$

for any $x \in [1-\xi(t), 1]$ and $t \in [\eta, T)$. Moreover, by (3.2), (3.3), and (3.5), we have

$$
(1-\epsilon)\epsilon^m u^{\gamma+m+1-\alpha}(1,t) \le y(t) \le (1-\epsilon)u^{\gamma+m+1-\alpha}(1,t), \qquad (3.6)
$$

or equivalently,

$$
c_1 u(1,t) \le y^{1/(\gamma+m+1-\alpha)}(t) \le c_2 u(1,t) \qquad \text{for } t \in [\eta, T). \qquad (3.7)
$$

We now show that $I(t) \geq 0$ on $[\eta, T)$.

$$\begin{aligned} I(t) &= (1-\alpha)\int_{1-\xi(t)}^{1}(u^m(x,t)-u^m(1-\xi(t),t))dx \\ &\quad +(\alpha-\gamma-1)\int_{1-\xi(t)}^{1}u^m(x,t)dx \\ &= (1-\alpha)\int_{1-\xi(t)}^{1}mu^{m-1}(\zeta(t),t)u_x(\zeta(t),t)(x+\xi(t)-1)dx \qquad (3.8)\\ &\quad +(\alpha-\gamma-1)\int_{1-\xi(t)}^{1}u^m(x,t)dx, \end{aligned}$$

where $1-\xi(t) < \zeta(t) < 1$. Because $\alpha, m > 1$, $\alpha-\gamma-1 > 0$, $u_x$, $u_{xx} \geq 0$, and $\int_{1-\xi(t)}^{1}(x+\xi(t)-1)dx = (1/2)(1-\epsilon)^2u^{2-2\alpha}(1,t)$, we can see that

$$I(t) \geq (1-\epsilon)\left[\frac{1}{2}m(1-\alpha)(1-\epsilon)+(\alpha-\gamma-1)\epsilon^m\right]u^{m+1-\alpha}(1,t) \geq 0. \qquad (3.9)$$

From (3.4), (3.7), and (3.9), it then follows that

$$y'(t) \leq u^{\gamma+\alpha}(1,t) \leq c_1^{-\gamma-\alpha}y^{(\gamma+\alpha)/(\gamma+m+1-\alpha)}(t), \qquad (3.10)$$

since $\gamma+\alpha > 2\alpha-m-1 > 0$. Integrating the above inequality from $t$ to $T$, we obtain

$$y^{(m+1-2\alpha)/(\gamma+m+1-\alpha)}(t) \leq \frac{2\alpha-m-1}{\gamma+m+1-\alpha}c_1^{-\gamma-\alpha}(T-t),$$

that is,

$$y^{1/(\gamma+m+1-\alpha)}(t) \geq c_3(T-t)^{1/(m+1-2\alpha)},$$

which in conjunction with (3.7) leads to (3.1).

Based on the general idea of [7], we then locate the blow-up point for the case $(m+1)/2 < \alpha \leq m$.

**Theorem 3.2.** *Suppose that condition* (H) *holds. Then for* $(m+1)/2 < \alpha \leq m$, *blow-up occurs only at* $x = 1$.

*Proof.* Introduce a function $w(x) \in C^2([0,1])$ defined by

$$w(x) = \begin{cases} h(x) & 0 \leq x \leq \frac{1}{2}, \\ (1-x)^2 & \frac{1}{2} < x \leq 1 \end{cases} \qquad (3.11)$$

with $w'(0) = 0$ and $w'(x) \leq 0$ for $0 < x \leq 1$. Let $v(x,t) = C_0[w(x)+B(T-t)]^{-\rho}$, where $C_0$ and $B$ are positive constants to be determined, and $\rho = 1/(2\alpha-m-1)$. Then $v(x,t)$ satisfies

$$mu^{m-1}v_t - v_{xx} = C_0\rho[w(x)+B(T-t)]^{-\rho-1}J(x,t) \qquad \text{in } (0,1)\times(0,T). \qquad (3.12)$$

Here $J(x,t) = Bmu^{m-1} - (\rho+1)(w'(x))^2/[w(x)+B(T-t)] + w''(x)$. Since $u_0(x) \geq \delta > 0$ and $u_t \geq 0$, for $(x,t) \in (1/2,1)\times(0,T)$,

$$\begin{aligned} J(x,t) &= Bmu^{m-1} - \frac{4(\rho+1)(1-x)^2}{(1-x)^2+B(T-t)} + 2 \\ &\geq Bm\delta^{m-1} - 4\rho - 2 \geq 0 \qquad (3.13) \end{aligned}$$

if $B \geq (4\rho+2)/m\delta^{m-1}$; for $(x,t) \in (0,1/2] \times (0,T)$,

$$J(x,t) \geq Bm\delta^{m-1} - \frac{(\rho+1)h_1^2}{h(\frac{1}{2})} - h_2 \geq 0 \tag{3.14}$$

if $B \geq [(\rho+1)h_1^2/h(1/2) + h_2]/m\delta^{m-1}$, where $h_1 = \max_{0\leq x\leq \frac{1}{2}} |h'(x)|$ and $h_2 = \max_{0\leq x\leq\frac{1}{2}} |h''(x)|$.

On the parabolic boundary, $v_x(0,t) = u_x(0,t) = 0$. By (1.3), if $C_0 \geq B^\rho C$, then $v(1,t) = C_0 B^{-\rho}(T-t)^{-\rho} \geq u(1,t)$; and when $t = 0$, $C_0$ can be chosen so large that

$$v(x,0) \geq C_0[w(0) + BT]^{-\rho} \geq \max_{0\leq x\leq 1} u_0(x) \geq u_0(x).$$

Then the maximum principle implies

$$u(x,t) \leq v(x,t) \qquad \text{for } (x,t) \in [0,1]\times[0,T),$$

which shows that for any $x \in [0,1)$,

$$u(x,t) \leq C_0 w^{-\rho}(x) < \infty.$$

Thus the proof is completed.

## References

[1] K. Deng, The blow–up behavior of the heat equation with Neumann boundary conditions, *J. Math. Anal. Appl.* **188** (1994), 641–650. MR **95i:**35120

[2] K. Deng and M. Xu, On solutions of a singular diffusion equation, preprint.

[3] M. Fila and H.A. Levine, Quenching on the boundary, *Nonlinear Anal. TMA* **21** (1993), 795–802. MR **95b:**35028

[4] M. Fila and P. Quittner, The blow–up rate for the heat equation with a non–linear boundary condition, *Math. Methods Appl. Sci.* **14** (1991), 197–205. MR **92a:**35023

[5] J. Filo, Diffusivity versus absorption through the boundary, *J. Differential Equations* **99** (1992), 281–305. MR **94d:**35083

[6] J.L. Gómez, V. Márquez, and N. Wolanski, Blow up results and localization of blow up points for the heat equation with a nonlinear boundary condition, *J. Differential Equations* **92** (1991), 384–401. MR **92j:**35098

[7] B. Hu and H.-M. Yin, The profile near blow-up time for solutions of the heat equation with a nonlinear boundary condition, *Trans. Amer. Math. Soc.* **346** (1994), 117–135. MR **95c:**35040

[8] N. Wolanski, Global behavior of positive solutions to nonlinear diffusion problems with nonlinear absorption through the boundary, *SIAM J. Math. Anal.* **24** (1993), 317–326. MR **93j:**35023

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