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A UNIVERSAL FUNCTIONAL EQUATION

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ABSTRACT. It is shown that the S-chains solving Rubel's universal fourth-order differential equation also satisfy a third-order functional equation.

1. Introduction and statement of results

In 1981, L. A. Rubel [3] published the following result concerning the approximation of continuous functions by solutions of algebraic differential equations (ADE):

There exists a nontrivial fourth-order ADE such that any real continuous function on the real axis can be uniformly approximated by the C^∞-solutions of this ADE. One such specific ADE is

$$3y'^4y''y''''^2 - 4y'^4y'''^2y'''' + 6y'^3y''^2y'''y'''' + 24y'^2y''^4y'''' - 12y'^3y''y'''^3 - 29y'^2y''^3y'''^2 + 12y''^7 = 0. \tag{1.1}$$

The basic idea in proving this theorem is to investigate the function

$$y = A \cdot f(\alpha t + \beta) + B \qquad (-1 \le \alpha t + \beta \le 1), \tag{1.2}$$

where A, B, α and β are real parameters, and where the function f is given by

$$f(s) := \int_{-1}^{s} e^{-1/(1-u^2)}\, du \qquad (-1 \le s \le 1). \tag{1.3}$$

The graph of f is called a *primitive S-module.* A survey on the theory of universal formulae and universal differential equations can be found in [1]. It seems to be obvious that the degree of the ADE given in (1.1) is best-possible. By using different S-modules constructed with

$$\int_{-\pi/2}^{s} e^{-1/\cos u}\, du \qquad \left(-\frac{\pi}{2} \le s \le \frac{\pi}{2}\right),$$

the author [2] gets a third-order ADE with coefficients depending on a certain ε-neighborhood and on the Lipschitz-class of the continuous functions.

Definition 1.1. Let D be any set of real numbers, and let $P(u_1, \dots, u_m)$ be a polynomial in $u_1, \dots, u_m$ with integer coefficients, which does not vanish identically. A real function $y = y(x)$ with domain D is said to be a local solution of a functional

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equation of degree $n \geq 0$ (with respect to the polynomial P), if for every point x_0 from D there are distinct rationals $r_1, \ldots, r_k$ and some $\varepsilon > 0$, such that

$$P(y(x+r_1), \ldots, y(x+r_k), y'(x+r_1), \ldots, y'(x+r_k), \ldots, y^{(n)}(x+r_k)) = 0$$

holds for every x with $x_0 \leq x \leq x_0 + \varepsilon$ and $x + r_\kappa \in D$ $(1 \leq \kappa \leq k)$.

Now we shall show that the functions from (1.2) are local solutions of a functional equation of degree 3, and that the corresponding polynomial P can be computed explicitly.

Theorem 1.1. *Every real continuous function F on the real axis can be uniformly approximated by C^∞-functions, which are all local solutions of a functional equation of degree* 3*. The corresponding polynomial $P(u_1, \ldots, u_9)$ has* 206 *terms, its greatest coefficient is* 358 318 080 000*, and it does not depend on F.*

The polynomial P from this theorem can be simplified in a certain sense. Put

$$u := \frac{y_1' y_1'''}{y_1''^2}, \quad v := \frac{y_2' y_2'''}{y_2''^2}, \quad w := \frac{y_3' y_3'''}{y_3''^2}, \tag{1.4}$$

where

$$y_i := y\left(x + \frac{i-1}{n}\right) \qquad (i = 1, 2, 3;\ n \in \mathbb{Z}_{>0}). \tag{1.5}$$

Then P can be written as a polynomial in u, v and w of degree 8 with respect to each variable.

2. On Theorem 1.1: Existence proof of a functional equation of degree 3 and theoretical foundation of its computation

The approximating function Y of any continuous function F is given by "S-chains", which means a C^∞-function that consists of S-modules (1.2) pieced together. L. A. Rubel [3] has proved that F is approximated uniformly by certain S-chains Y of this type. So it remains to prove the existence of a functional equation, having every S-chain as a local solution.

Let x_0 be any real number. Obviously, for some sufficiently large positive integer n every real number x with $x_0 \leq x \leq x_0 + 1/n$ belongs to the domain of some function

$$y = A \cdot f(\alpha x + \beta) + B \qquad (-1 \leq \alpha x + \beta \leq 1) \tag{2.1}$$

from (1.2) and (1.3), which represents a certain section from the approximating function Y. Let $\alpha \geq 0$ first. Without loss of generality, we may assume that

$$-1 \leq \alpha x_0 + \beta < 1;$$

hence, for sufficiently large n, we have

$$\alpha x_0 + \beta < 1 - \frac{3\alpha}{n}.$$

By $x_0 \leq x \leq x_0 + 1/n$, it follows that

$$-1 \leq \alpha x + \beta < 1 - \frac{2\alpha}{n},$$

or

$$-1 + \frac{2\alpha}{n} \leq \alpha x + \beta + \frac{2\alpha}{n} < 1.$$

For brevity, we define

$$s_i := \alpha x + \beta + \frac{\alpha(i-1)}{n} \qquad (i = 1, 2, 3)$$

for any x with $x_0 \le x \le x_0 + 1/n$, which means that $-1 \le s_i < 1$ for $i = 1, 2, 3$. Thus, passing from x to $x + 1/n$ and to $x + 2/n$, we do not leave the domain of the function y in (2.1). If $\alpha < 0$, we may assume that $1 \ge \alpha x_0 + \beta > -1$ and proceed in the same way. From

$$1 \ge \alpha x + \beta > -1 - \frac{2\alpha}{n}$$

we conclude that $1 \ge s_i > -1$ $(i = 1, 2, 3)$.

First we treat the case where $y_1'' y_2'' y_3''(x) \ne 0$ with y_i from (1.5). Differentiating f with respect to s, we obtain from (1.3) that

$$f'(s) = e^{-1/(1-s^2)}, \qquad f''(s) = \frac{-2s}{(1-s^2)^2} e^{-1/(1-s^2)}, \qquad f'''(s) = \frac{6s^4 - 2}{(1-s^2)^4} e^{-1/(1-s^2)}. \tag{2.2}$$

Note that $f'(\pm 1) = 0$, $f''(\pm 1) = 0$ and $f'''(\pm 1) = 0$ do exist. Differentiating the functions y_i from (1.5) with respect to x, we have, using (2.1),

$$y_i' = \alpha A f'(s_i), \quad y_i'' = \alpha^2 A f''(s_i), \quad y_i''' = \alpha^3 A f'''(s_i) \qquad (i = 1, 2, 3). \tag{2.3}$$

To make the desired functional equation independent from special values of A, B, α and β, we have to eliminate these parameters. Thus, we get for every $i = 1, 2, 3$:

$$\begin{aligned} y_i' y_i''' &= \alpha^4 A^2 f'(s_i) f'''(s_i) = \frac{(6s_i^4 - 2)\alpha^4 A^2}{(1 - s_i^2)^4} e^{-2/(1-s_i^2)} \\ &= \frac{3s_i^4 - 1}{2s_i^2} y_i''^2 \quad \text{(by (2.2))}. \end{aligned}$$

From (1.4), we conclude for $y_i'' \ne 0$:

$$u = \frac{3s_1^4 - 1}{2s_1^2}, \quad v = \frac{3s_2^4 - 1}{2s_2^2}, \quad w = \frac{3s_3^4 - 1}{2s_3^2}.$$

In what follows, we work with w; there are no essential differences in using u or v. Since $s_3^2 \ge 0$, we can solve the equation for s_3^2:

$$s_3^2 = \frac{w}{3} + \sqrt{\left(\frac{w}{3}\right)^2 + \frac{1}{3}}.$$

Let σ_3 denote the sign of s_3; hence

$$s_3 = \sigma_3 \sqrt{\frac{w}{3} + \sqrt{\left(\frac{w}{3}\right)^2 + \frac{1}{3}}}.$$

Note that $\sigma_3 \in \{-1, 1\}$ depends on x. It is $s_3 \ne 0$, since otherwise $f''(s_3)$ and y_3'' vanish by (2.2) and (2.3). But this contradicts our assumption on y_3''. We also have

$$s_2 = \sigma_2 \sqrt{\frac{v}{3} + \sqrt{\left(\frac{v}{3}\right)^2 + \frac{1}{3}}} \qquad (\sigma_2 = \operatorname{sign} s_2 \in \{-1, 1\})$$

and

$$s_1 = \sigma_1 \sqrt{\frac{u}{3} + \sqrt{\left(\frac{u}{3}\right)^2 + \frac{1}{3}}} \qquad (\sigma_1 = \operatorname{sign} s_1 \in \{-1, 1\}).$$

There is a simple relation between s_1, s_2 and s_3, namely

$$2s_2 = 2\alpha x + 2\beta + \frac{2\alpha}{n} = s_1 + s_3,$$

or

$$2s_1 s_3 = 4s_2^2 - s_1^2 - s_3^2.$$

Squaring this equation a second time, we get

$$4s_1^2 s_3^2 = (4s_2^2 - s_1^2 - s_3^2)^2.$$

Putting in the expressions for s_1, s_2 and s_3 and multiplying by 9, we get an algebraic relation between u, v and w:

$$\begin{aligned} &4 \cdot \left(u + \sqrt{u^2+3}\right)\left(w + \sqrt{w^2+3}\right) \\ &= \left\{4\left(v + \sqrt{v^2+3}\right) - u - \sqrt{u^2+3} - w - \sqrt{w^2+3}\right\}^2 . \end{aligned} \tag{2.4}$$

This equation and (1.4) prove that every S-chain is a local solution of a certain functional equation of degree 3. Note that the cases where $y_1'' y_2'' y_3''(x) = 0$ do not belong to any essential exception, since one may argue with the continuity of all occurring functions. Therefore the functional equation holds for such x, too.

Similar to the proof of Rubel's universal differential equation, a surprising amount of cancellation takes place if we eliminate the square roots in (2.4) step by step. First we introduce some useful notations:

$$a := u^2 + 3, \quad b := v^2 + 3, \quad c := w^2 + 3;$$
$$A := 4v - u + w, \quad B := -4(4v - u - w), \quad C := 4v + u - w,$$
$$D := uw + 4uv + 4vw - u^2 - 16v^2 - w^2 - 27.$$

Hence, (2.4) takes the form

$$A\sqrt{a} + B\sqrt{b} + C\sqrt{c} = -\left(4\sqrt{ab} + \sqrt{ac} + 4\sqrt{bc} + D\right).$$

Squaring both sides, we get

$$E\sqrt{ab} + F\sqrt{ac} = G\sqrt{bc} + H, \tag{2.5}$$

where

$$\begin{aligned} E &:= 2AB - 8D - 8c, \\ F &:= 2AC - 2D - 32b, \\ G &:= 8D + 8a - 2BC, \\ H &:= D^2 + 16ab + ac + 16bc - aA^2 - bB^2 - cC^2. \end{aligned}$$

Squaring again, we conclude from (2.5) that

$$4(aEF - GH)^2 bc = (bcG^2 + H^2 - abE^2 - acF^2)^2.$$

The final computation is left to a computer. Putting in all the expressions for $A, B, \ldots, H$ and for a, b, c, we get the polynomial P described in Theorem 1.1.

Note added in proof

In the context of Theorem 1.1 one may replace Definition 1.1 by the following more convenient definition: A real function $y = y(x)$ with domain $\mathbb{R}$ is said to be *a local solution of a functional equation of degree* 3, if there is a polynomial $P(u_1, \dots, u_9)$ such that for every point x_1 from $\mathbb{R}$ and for some sufficiently small $\varepsilon > 0$ depending on x_1 the identity

$$P\left(y'(x_1), y'\left(\frac{x_1+x_2}{2}\right), y'(x_2), y''(x_1), y''\left(\frac{x_1+x_2}{2}\right), \dots, y'''(x_2)\right) = 0$$

holds for every real x_2 with $0 \leq x_2 - x_1 \leq \varepsilon$.

Instead of (1.5) put:

$$y_1 := y(x_1), \quad y_2 := y\left(\frac{x_1+x_2}{2}\right), \quad y_3 := y(x_2).$$

Then, Theorem 1.1 remains true, and there are no essential changes within the proof.

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