PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 1, January 1999, Pages 205–210
S 0002-9939(99)05030-3

SOME REMARKS ON THE REAL RANK OF NON-UNITAL C*-ALGEBRAS

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(Communicated by David R. Larson)

Abstract. For a non-unital C*-algebra A, let $A^\sim$ be the C*-algebra obtained from A by adjoining an identity. In this paper we show that

$$\mathrm{RR}(C_0(X)\otimes A)=\mathrm{RR}(C_0(X)\otimes A^\sim),$$

where X is a locally compact Hausdorff space with $\mathrm{RR}(C_0(X))\le 1$.

1. Introduction

For a unital C*-algebra A, L.G. Brown and G.K. Pedersen ([2]) define the real rank of A, denoted by $\mathrm{RR}(A)$, as the least integer n such that for each $(n+1)$-tuple $(x_0,\cdots,x_n)$ of self-adjoint elements in A and every $\varepsilon>0$, there exists an $(n+1)$-tuple $(y_0,\cdots,y_n)$ in $A_{s.a}$ such that $\sum_{j=0}^n y_j^2$ is invertible and

$$\|x_k-y_k\|<\varepsilon \qquad (k=0,\dots,n).$$

M.A. Rieffel ([5]) defines the stable rank of A, denoted by $\mathrm{sr}(A)$, as the least integer n such that for each n-tuple $(x_1,\cdots,x_n)$ of elements in A and every $\varepsilon>0$, there exists an n-tuple $(y_1,\cdots,y_n)$ in A such that $\sum_{j=1}^n y_j^*y_j$ is invertible and

$$\|x_k-y_k\|<\varepsilon \qquad (k=1,\dots,n).$$

If A is non-unital, then we define the real rank of A and the stable rank of A by $\mathrm{RR}(A^\sim)$ and $\mathrm{sr}(A^\sim)$ respectively, where the C*-algebra $A^\sim$ is obtained from A by adjoining an identity.

In [2], L.G. Brown and G.K. Pedersen show that

$$\mathrm{RR}(C(X))=\dim X$$

for a unital commutative C*-algebra $C(X)$ of all the continuous functions on a compact Hausdorff space X, where $\dim X$ means the covering dimension of X. For every compact space X and every paracompact Y, not both empty, we have

$$\dim(X\times Y)=\dim X+\dim Y$$

(see [3]). In [4], M. Nagisa, H. Osaka and N.C. Phillips show that

$$\mathrm{RR}(C_0(X)\otimes A)\le \dim X+\mathrm{RR}(A)$$

Received by the editors May 9, 1997.
1991 *Mathematics Subject Classification.* Primary 46L05.
Key words and phrases. C*-algebra, real rank, stable rank.

for any C*-algebra and any locally compact, σ-compact, Hausdorff space, where $C_0(X)$ means all the continuous functions vanishing at infinity on X. We remark that a pathological phenomena occurs without the assumption of countability. Strictly speaking, for a non-unital C*-algebra A, they use the fact $\mathrm{RR}(C_0(X)\otimes A) \leq \mathrm{RR}(C_0(X)\otimes A^{\sim})$ and show that

$$\mathrm{RR}(C_0(X)\otimes A^{\sim}) \leq \dim X + \mathrm{RR}(A).$$

In this note, we consider whether $\mathrm{RR}(C_0(X)\otimes A)$ differs from $\mathrm{RR}(C_0(X)\otimes A^{\sim})$ or not. Our result is

$$\mathrm{RR}(C_0(X)\otimes A) = \mathrm{RR}(C_0(X)\otimes A^{\sim})$$

for any non-unital C*-algebra A and any locally compact Hausdorff space X with $\mathrm{RR}(C_0(X)) \leq 1$.

2. Main result

We shall denote the one-point compactification of a locally compact Hausdorff space X by $X\cup\{\infty\}$; then we can identify $C_0(X)^{\sim}$ and $C(X\cup\{\infty\})$. We shall also denote, by $C_0(X)\otimes A$, all the A-valued continuous functions vanishing at infinity on X. Then we can canonically identify $C_0(X, A)$ and the C*-tensor product $C_0(X)\otimes A$ of $C_0(X)$ and a C*-algebra A. We use q_A to denote the natural quotient map $A^{\sim} \ni a+\lambda 1 \mapsto \lambda \in \mathbb{C} \cong A^{\sim}/A$.

The following lemma is shown in [2], [6].

Lemma 1. *Let* $0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$ *be an exact sequence of C*-algebras. Then* $\mathrm{RR}(A) = 0$ *if and only if* $\mathrm{RR}(J) = \mathrm{RR}(A/J) = 0$ *and any projection in* A/J *can be lifted to a projection in* A.

The following proposition is essentially proved in [4]. Here, we extend the original one to the case that a C*-algebra A is non-unital and a space X, instead of the interval, is general.

Proposition 2. *Let* X *be a locally compact Hausdorff space with* $\mathrm{RR}(C_0(X)) \geq 1$. *Then* $\mathrm{RR}(C_0(X)\otimes A) \geq 1$ *for any* C^**-algebra* A.

Proof. For any $x \in X$, the map $C_0(X)\otimes A\ (\simeq C_0(X, A)) \ni f \mapsto f(x) \in A$ is the surjective *-homomorphism. This implies that $\mathrm{RR}(C_0(X)\otimes A) \geq \mathrm{RR}(A)$. So we may assume that $\mathrm{RR}(A) = 0$, and we consider the following exact sequence of C^*-algebras:

$$0 \longrightarrow C_0(X)\otimes A \longrightarrow C(X\cup\{\infty\})\otimes A \xrightarrow{\varphi} A \longrightarrow 0,$$

where φ is defined by $\varphi(f) = f(\infty)$ for all $f \in C(X\cup\{\infty\})\otimes A$. Clearly any projection in A can be lifted to a projection in $C(X\cup\{\infty\})\otimes A$. This means that $\mathrm{RR}(C(X\cup\{\infty\})\otimes A) = 0$, if $\mathrm{RR}(C_0(X)\otimes A) = 0$. So it is enough to show that $\mathrm{RR}(C(X)\otimes A) \geq 1$ for a compact Hausdorff space X.

From the assumption $\mathrm{RR}(C(X)) \geq 1$, there exists a self-adjoint element f in $C(X)$ and a positive number δ such that, if g is in $C(X)_{s.a.}$ with $\|f-g\| < \delta$, then $g(y) = 0$ for some $y \in X$. Since X is compact, we can find a positive constant K such that $f(x) > -K$ for any $x \in X$. Let a be a positive element in A with $\|a\| = 1$. We define a self-adjoint element $F \in (C(X)\otimes A)^{\sim}$ by

$$F(x) = (f(x)+K)a - K1 \qquad (x \in X),$$

where we identify $(C(X) \otimes A)^\sim$ and the set $\{g \in C(X, A^\sim) \mid q_A(g(\cdot))$ is a constant function on $X\}$. Let us denote, for any element b in A, the set $\{\lambda \in \mathbb{C} \mid b - \lambda 1$ is not invertible$\}$ by the symbol $Sp(b)$. Then $f(x) = \max Sp(F(x))$. In fact, by the positivity of $F(x) + K1$, we have

$$\begin{aligned}\max Sp(F(x)) + K &= \max Sp(F(x) + K1) = \|F(x) + K1\| \\ &= \|(f(x) + K)a\| = f(x) + K.\end{aligned}$$

If G is a self-adjoint element in $(C(X) \otimes A)^\sim$ with $\|F - G\| < \delta$, then we have $\max Sp(G(\cdot)) \in C(X)_{s.a.}$ and $\|f - \max Sp(G(\cdot))\| < \delta$. Therefore $\max Sp(G(y)) = 0$ for some $y \in X$, that is, G is not invertible. This implies $\mathrm{RR}(C(X) \otimes A) \geq 1$. $\square$

Theorem 3. *Let X be a locally compact Hausdorff space with $\mathrm{RR}(C_0(X)) \leq 1$. Then $\mathrm{RR}(C_0(X) \otimes A) = \mathrm{RR}(C_0(X) \otimes A^\sim)$ for any non-unital C*-algebra A.*

Proof. We shall prove only the case that X is non-compact. We can similarly prove the case that X is compact.

For a locally compact, non-compact Hausdorff space X, we can identify $(C_0(X) \otimes A^\sim)^\sim$ and

$$\{F \in C(X \cup \{\infty\}, A^\sim) \mid F(\infty) = q_A(F(\infty))1\}.$$

In this identification, we can see $(C_0(X) \otimes A)^\sim$ as

$$\{F \in C(X \cup \{\infty\}, A^\sim) \mid F(\infty) = q_A(F(x))1, \quad \text{for all } x \in X\}.$$

We consider the following split exact sequence of C*-algebras:

$$0 \longrightarrow C_0(X) \otimes A \longrightarrow (C_0(X) \otimes A^\sim)^\sim \underset{j}{\overset{\pi}{\longleftrightarrow}} C(X \cup \{\infty\}) \longrightarrow 0,$$

where $\pi(F)(x) = q_A(F(x))$ and $j(f)(x) = f(x)1$ for $F \in (C_0(X) \otimes A^\sim)^\sim$, $f \in C_0(X \cup \{\infty\})$ and $x \in X \cup \{\infty\}$. Clearly we have

$$\mathrm{RR}(C_0(X) \otimes A) \leq \mathrm{RR}(C_0(X) \otimes A^\sim).$$

At first, we examine the case that the real rank of $C_0(X) \otimes A$ is zero. By Proposition 2, we have $\mathrm{RR}(C_0(X)) = 0$ and it is clear that any projection in $C(X \cup \{\infty\})$ can be lifted to a projection in $(C_0(X) \otimes A^\sim)^\sim$. So we get that $\mathrm{RR}((C_0(X) \otimes A^\sim)) = 0$ by Lemma 1.

In the case that $\mathrm{RR}(C_0(X) \otimes A) = n \geq 1$, we shall show that $\mathrm{RR}(C_0(X) \otimes A^\sim) \leq n$. Let $F_0, F_1, \ldots, F_n$ be self-adjoint elements in $(C_0(X) \otimes A^\sim)^\sim$, and set

$$g_0 = \pi(F_0),\ g_1 = \pi(F_1),\ f_2 = \pi(F_2), \ldots, f_n = \pi(F_n).$$

Since $\mathrm{RR}(C(X \cup \{\infty\})) \leq 1$, for any $\varepsilon > 0$ we can choose $f_0, f_1 \in C(X \cup \{\infty\})_{s.a.}$ such that $\|g_0 - f_0\| < \varepsilon$, $\|g_1 - f_1\| < \varepsilon$ and $f_0^2 + f_1^2 > 0$. We consider an element Δ in $\mathbb{M}_{n+1}(C(X \cup \{\infty\}))$ as follows:

$$\Delta = \begin{pmatrix} \frac{f_0}{k_0} & \frac{f_1}{k_0} & \frac{f_2}{k_0} & \frac{f_3}{k_0} & \cdots & \frac{f_{n-1}}{k_0} & \frac{f_n}{k_0} \\ -\frac{f_1}{k_1} & \frac{f_0}{k_1} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{f_0 f_2}{k_2} & -\frac{f_1 f_2}{k_2} & \frac{f_0^2 + f_1^2}{k_2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{f_0 f_n}{k_n} & -\frac{f_1 f_n}{k_n} & -\frac{f_2 f_n}{k_n} & -\frac{f_3 f_n}{k_n} & \cdots & -\frac{f_{n-1} f_n}{k_n} & \frac{\sum_{k=0}^{n-1} f_k^2}{k_n} \end{pmatrix},$$

where $k_0 = \sqrt{f_0^2 + f_1^2 + \cdots + f_n^2}$, $k_1 = \sqrt{f_0^2 + f_1^2}$ and

$$k_l = \sqrt{(f_0^2 + \cdots + f_{l-1}^2)(f_0^2 + \cdots + f_l^2)} \qquad (2 \leq l \leq n).$$

Then we have

$$^t\Delta\Delta = I_{n+1} \quad \text{and} \quad \frac{1}{k_0}\Delta \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We define self-adjoint elements $G_0, G_1, \ldots, G_n$ in $(C_0(X) \otimes A^\sim)^\sim$ by the following relation:

$$(j \otimes id_{n+1})(\frac{1}{k_0}\Delta) \begin{pmatrix} F_0 - j(g_0) + j(f_0) \\ F_1 - j(g_1) + j(f_1) \\ F_2 \\ \vdots \\ F_n \end{pmatrix} = \begin{pmatrix} G_0 \\ G_1 \\ G_2 \\ \vdots \\ G_n \end{pmatrix}.$$

Then we can easily check

$$q_A(G_0(x)) = 1,\ q_A(G_1(x)) = 0, \ldots, q_A(G_n(x)) = 0$$

for any $x \in X \cup \{\infty\}$, that is, $G_0, \ldots, G_n$ belong to $(C_0(X) \otimes A)^\sim$. So we can choose self-adjoint elements $\tilde{G_0}, \tilde{G_1}, \ldots, \tilde{G_n}$ in $(C_0(X) \otimes A)^\sim$ satisfying $\|G_l - \tilde{G_l}\| < (1/\|k_0\|)\varepsilon$ for $0 \leq l \leq n$, and $\sum_{l=0}^n (\tilde{G_l})^2$ is invertible. If we put

$$\begin{pmatrix} \tilde{F_0} \\ \tilde{F_1} \\ \vdots \\ \tilde{F_n} \end{pmatrix} = (j \otimes id_{n+1})(k_0\, {}^t\Delta) \begin{pmatrix} \tilde{G_0} \\ \tilde{G_1} \\ \vdots \\ \tilde{G_n} \end{pmatrix},$$

then we get $\tilde{F_l} \in (C_0(X) \otimes A^\sim)^\sim$ with $\|F_l - \tilde{F_l}\| < \varepsilon$ for $0 \leq l \leq n$, and $\sum_{l=0}^n (\tilde{F_l})^2 = j(k_0^2) \cdot \sum_{l=0}^n (\tilde{G_k})^2$ is invertible. □

In the above theorem, for a space X with $\mathrm{RR}(C_0(X)) \geq 2$, it is not true that $\mathrm{RR}(C_0(X) \otimes A) = \mathrm{RR}(C_0(X) \otimes A^\sim)$ in general. For example, let us denote by I the $[0, 1]$-interval and by $\mathbb{K}$ the C*-algebra of all the compact operators. Then we have

$$\mathrm{RR}(C(I \times I) \otimes \mathbb{K}) = 1$$

(see [1], [2]). But, by the fact $C(I \times I) \otimes \mathbb{K}^\sim / C(I \times I) \otimes \mathbb{K} \cong C(I \times I)$, we have

$$\mathrm{RR}(C(I \times I) \otimes \mathbb{K}^\sim) \geq 2.$$

We can apply the above argument for real rank to that for stable rank. Then we have the following statement:

Proposition 4. *Let X be a locally compact Hausdorff space, and A a non-unital C*-algebra.*

(1) *If* $\mathrm{sr}(C_0(X)) = 1$, *then*

$$\mathrm{sr}(C_0(X) \otimes A) = \mathrm{sr}(C_0(X) \otimes A^\sim).$$

(2) *If* $\mathrm{sr}(C_0(X)) = 2$ *and* $\mathrm{sr}(C_0(X) \otimes A) \geq 2$, *then*

$$\mathrm{sr}(C_0(X) \otimes A) = \mathrm{sr}(C_0(X) \otimes A^{\sim}).$$

Proof. We use the identification as stated in the proof of Theorem 3. From the split exact sequence of C*-algebras

$$0 \longrightarrow C_0(X) \otimes A \longrightarrow (C_0(X) \otimes A^{\sim})^{\sim} \underset{j}{\overset{\pi}{\longleftrightarrow}} C(X \cup \{\infty\}) \longrightarrow 0,$$

we have $\mathrm{sr}(C_0(X) \otimes A) \leq \mathrm{sr}(C_0(X) \otimes A^{\sim})$.

First, we examine the case that $\mathrm{sr}(C_0(X)) \leq 2$ and $\mathrm{sr}(C_0(X) \otimes A) \geq 2$. Suppose that $\mathrm{sr}(C_0(X) \otimes A) = n$. We shall show that $\mathrm{sr}(C_0(X) \otimes A^{\sim}) \leq n$. Let $F_1, F_2, \ldots, F_n$ be elements in $(C_0(X) \otimes A^{\sim})^{\sim}$, and set

$$g_1 = \pi(F_1),\ g_2 = \pi(F_2),\ f_3 = \pi(F_3), \ldots, f_n = \pi(F_n).$$

Since $\mathrm{sr}(C(X \cup \{\infty\})) \leq 2$, for any $\varepsilon > 0$ we can choose $f_1, f_2 \in C(X \cup \{\infty\})$ such that $\|g_1 - f_1\| < \varepsilon$, $\|g_2 - f_2\| < \varepsilon$ and $|f_1|^2 + |f_2|^2 > 0$. We consider an element Δ in $\mathbb{M}_n(C(X \cup \{\infty\}))$ as follows:

$$\Delta = \begin{pmatrix} \frac{\overline{f_1}}{k_1} & \frac{\overline{f_2}}{k_1} & \frac{\overline{f_3}}{k_1} & \frac{\overline{f_4}}{k_1} & \cdots & \frac{\overline{f_{n-1}}}{k_1} & \frac{\overline{f_n}}{k_1} \\ -\frac{\overline{f_2}}{k_2} & \frac{\overline{f_1}}{k_2} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{\overline{f_1} f_3}{k_3} & -\frac{\overline{f_2} f_3}{k_3} & \frac{|f_1|^2+|f_2|^2}{k_3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{\overline{f_1} f_n}{k_n} & -\frac{\overline{f_2} f_n}{k_n} & -\frac{\overline{f_3} f_n}{k_n} & -\frac{\overline{f_4} f_n}{k_n} & \cdots & -\frac{\overline{f_{n-1}} f_n}{k_n} & \frac{\sum_{k=1}^{n-1} |f_k|^2}{k_n} \end{pmatrix},$$

where $\overline{f}$ means the complex conjugate of f,

$$k_1 = \sqrt{|f_1|^2 + \cdots + |f_n|^2}, \qquad k_2 = \sqrt{|f_1|^2 + |f_2|^2}$$

and

$$k_l = \sqrt{(|f_1|^2 + \cdots + |f_{l-1}|^2)(|f_1|^2 + \cdots + |f_l|^2)} \qquad (3 \leq l \leq n).$$

Then we have

$${}^t\overline{\Delta}\Delta = I_n \quad \text{and} \quad \frac{1}{k_1}\Delta \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

For elements $G_1, G_2, \ldots, G_n$ in $(C_0(X) \otimes A)^{\sim}$ which are defined by

$$(j \otimes id_n)(\frac{1}{k_1}\Delta) \begin{pmatrix} F_1 - j(g_1) + j(f_1) \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ \vdots \\ G_n \end{pmatrix},$$

we can choose elements $\tilde{G_1}, \tilde{G_2}, \ldots, \tilde{G_n}$ in $(C_0(X) \otimes A)^{\sim}$ satisfying $\|G_l - \tilde{G_l}\| < (1/\|k_1\|)\varepsilon$ for $1 \leq l \leq n$, and $\sum_{l=1}^{n} (\tilde{G_l})^2$ is invertible. Then we can get the desired approximants $\tilde{F_1}, \dots , \tilde{F_n}$ by the relation

$$\begin{pmatrix} \tilde{F_1} \\ \tilde{F_2} \\ \vdots \\ \tilde{F_n} \end{pmatrix} = (j \otimes id_n)(k_1 {}^t\overline{\Delta}) \begin{pmatrix} \tilde{G_1} \\ \tilde{G_2} \\ \vdots \\ \tilde{G_n} \end{pmatrix} .$$

Next, we consider the case that $\mathrm{sr}(C_0(X) \otimes A) = 1$ and $\mathrm{sr}(C_0(X)) = 1$. The above argument can be applied to this case. There, Δ is a 1×1 matrix $\frac{f}{|f|}$ for an invertible element f in $C(X \cup \{\infty\})$. $\square$

We also remark that

$$\mathrm{sr}(C(I^4) \otimes \mathbb{K}) = 2 \ \text{ and } \mathrm{sr}(C(I^4) \otimes \mathbb{K}^{\sim}) \geq \mathrm{sr}(C(I^4)) = 3.$$

So, in the case $\mathrm{sr}(C_0(X)) \geq 3$,

$$\mathrm{sr}(C_0(X) \otimes A) = \mathrm{sr}(C_0(X) \otimes A^{\sim})$$

does not hold in general.

In [4], the following fact is proved: if $\mathrm{sr}(C(I^n) \otimes A) = 1$ for a unital C*-algebra A, then we have $n = 0$ or 1 (i.e. $\mathrm{sr}(C(I^n)) = 1$). For a non-unital C*-algebra A and a locally compact Hausdorff space X, we do not know whether $\mathrm{sr}(C_0(X) \otimes A) = 1$ implies $\mathrm{sr}(C_0(X)) = 1$ or not.

ACKNOWLEDGEMENT

I would like to express my deep gratitude to M. Nagisa for his help with this paper.

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