

METRIZABLE AND \mathbb{R} -METRIZABLE BETWEENNESS SPACES

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(Communicated by Carl G. Jockusch, Jr.)

ABSTRACT. It is proved that the theory of the class of all betweenness spaces metrizable by real-valued metrics does not coincide with the theory of the class of all betweenness spaces metrizable by metrics taking values in any ordered field. This solves a problem raised by Mendris and Zlatoš.

Let d be a metric on a nonempty set A taking values in an ordered field F . The ternary betweenness relation T_d on A is defined by

$$T_d(x, y, z) \Leftrightarrow d(x, y) + d(y, z) = d(x, z).$$

A first-order structure (A, T) with a single ternary relation T is called a metrizable betweenness space if $T = T_d$ for some metric d on A ; it is called an \mathbb{R} -metrizable betweenness space if $T = T_d$ for some real-valued d .

As proved by R. Mendris and P. Zlatoš in [1], the class \mathcal{M} of all metrizable betweenness spaces, i.e., the class of all first-order structures of the form (A, T_d) , where d is a metric on A , is a universal elementary one. On the other hand, being not closed under elementary extensions, the class \mathcal{M}_0 of all \mathbb{R} -metrizable betweenness spaces is not elementary. However, the question of whether \mathcal{M} is the least elementary class containing \mathcal{M}_0 or, equivalently, the question of whether $\text{Th } \mathcal{M}_0 = \text{Th } \mathcal{M}$ remained open. In this short note we will answer that question negatively.

Let $|xy| \leq |uv|$ stand as the abbreviation for the formula

$$(T(x, y, v) \ \& \ T(u, x, y)) \vee (T(x, y, u) \ \& \ T(v, x, y)).$$

Obviously, in a metrizable betweenness space (A, T_d) , $|xy| \leq |uv|$ implies $d(x, y) \leq d(u, v)$ for any $x, y, u, v \in A$.

Further, let $\phi(x, y, z, u, v)$ be the formula expressing that x, y, z, u, v are five distinct elements, and $T(x, y, z), T(u, x, y), T(u, z, v)$ are the only non-trivial betweenness relations among them.

Finally, let us denote by θ the sentence:

$$\begin{aligned} (\exists x, y, z, u, v) \phi(x, y, z, u, v) \ \& \ (\forall x, y, z, u, v) [\phi(x, y, z, u, v) \\ \Rightarrow ((\exists z')(z' \neq v \ \& \ T(z, z', v) \ \& \ |xy| \leq |zz'|)]]. \end{aligned}$$

Theorem 1. $\neg\theta \in \text{Th } \mathcal{M}_0 \setminus \text{Th } \mathcal{M}$; consequently, the least elementary class containing \mathcal{M}_0 is a proper subclass of \mathcal{M} .

Received by the editors April 2, 1997 and, in revised form, May 15, 1997.

1991 *Mathematics Subject Classification.* Primary 03B30, 03C52, 51F99.

Key words and phrases. Betweenness relation, metric, ordered field, elementary class.

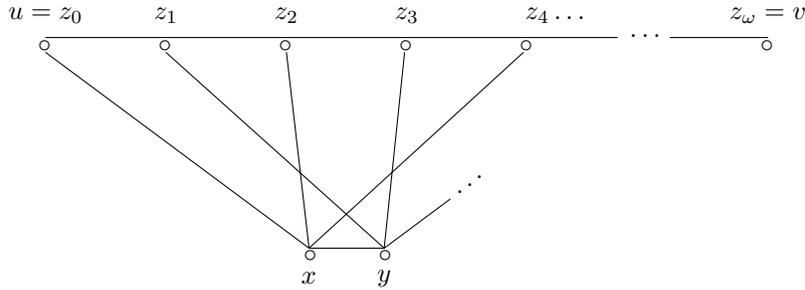


DIAGRAM 1

Proof. (1) $\neg\theta \in \text{Th } \mathcal{M}_0$: Let (A, T) be a metrizable betweenness space satisfying θ , let d be any metric on A inducing T , and let x, y, z, u, v be the elements of A guaranteed by θ . By induction one can construct a sequence $(z_n)_{n < \omega}$ in A , such that $z_0 = u, z_1 = z$ and $z_{n+1} = z'_n$ for $n \geq 1$, i.e., $z_{n+1} \neq v, T(z_n, z_{n+1}, v)$ and $|xy| \leq |z_n z_{n+1}|$. Then both $T(u, z_m, z_n)$ and $T(z_m, z_n, v)$ can easily be verified for any $m \leq n < \omega$. Therefore,

$$d(u, v) = d(u, z_1) + d(z_1, z_2) + \dots + d(z_{n-1}, z_n) + d(z_n, v) \geq n \cdot d(x, y)$$

for each $n < \omega$. As $d(x, y) > 0$, d cannot take values in an Archimedean ordered field.

(2) $\neg\theta \notin \text{Th } \mathcal{M}$: It suffices to find an $(A, T) \in \mathcal{M}$ satisfying θ . Let A consist of the chain $u = z_0 < z_1 < \dots < z_n < z_{n+1} < \dots < z_\omega = v$, ordered by the type $\omega + 1$, and two other elements x, y . Let T contain all the triples of the form (a, a, b) ($a, b \in A$), $(x, y, z_{2n+1}), (z_{2n}, x, y)$ ($n < \omega$), (z_m, z_n, z_α) ($m < n < \alpha \leq \omega$), and the reversed ones — see Diagram 1.

One can check that (A, T) satisfies θ . On the other hand, (A, T) is metrizable in any non-Archimedean ordered field F . Indeed, identifying $n < \omega$ with $n \cdot 1 \in F$ and ω with an arbitrary element of F bigger than any n , the metric $d : A \times A \rightarrow F$ can be defined by, say,

$$\begin{aligned} d(x, y) &= 1, & d(z_m, z_n) &= 2|m - n|, & d(z_n, v) &= \omega - 2n, \\ d(x, z_{2n}) &= d(y, z_{2n+1}) = d(x, v) = d(y, v) = \omega, & d(x, z_{2n+1}) &= d(y, z_{2n}) = \omega + 1, \end{aligned}$$

for all $m, n < \omega$. Then $T = T_d$ is plain. □

Thus we see that \mathcal{M} is not the least elementary class containing \mathcal{M}_0 . Nevertheless, \mathcal{M} is still a certain closure of \mathcal{M}_0 .

Theorem 2. \mathcal{M} is both the least universal class and the least universal-existential class containing \mathcal{M}_0 , i.e., $\text{Th}_\forall \mathcal{M}_0 = \text{Th}_{\forall\exists} \mathcal{M}_0 = \text{Th } \mathcal{M}$.

Proof. Let \mathcal{K} denote the least universal-existential class containing \mathcal{M}_0 . Obviously, $\mathcal{K} \subseteq \mathcal{M}$. It suffices to show that every countable $(A, T) \in \mathcal{M}$ belongs to \mathcal{K} . Each (A, T) can be written as the union of a chain of its finite substructures. Since \mathcal{M} and \mathcal{M}_0 contain the same finite structures and the universal-existential class \mathcal{K} is closed under chain unions, $(A, T) \in \mathcal{K}$. Hence $\text{Th}_{\forall\exists} \mathcal{M} = \mathcal{M}_0 = \text{Th } \mathcal{M}$. Since $\text{Th } \mathcal{M} \subset \text{Th } \mathcal{M}_0$, we have $\text{Th}_\forall \mathcal{M} \subseteq \text{Th}_\forall \mathcal{M}_0$, therefore, since $\text{Th}_\forall \mathcal{M} = \text{Th } \mathcal{M}$ (by part (ii) of the Theorem in [1]), $\text{Th } \mathcal{M} \subseteq \text{Th}_\forall \mathcal{M}_0$. The statement of the theorem now follows from $\text{Th}_\forall \mathcal{M}_0 \subseteq \text{Th}_{\forall\exists} \mathcal{M}_0 = \text{Th } \mathcal{M}$. □

As $\neg\theta$ is equivalent to an existential-universal sentence, the counterexample of Theorem 1 is the best possible.

REFERENCES

- [1] R. Mendris and P. Zlatoš, *Axiomatization and undecidability results for metrizable betweenness relations*, Proc. Amer. Math. Soc. **123** (1995), 873–882. MR **95d**:03008

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