

LENGTHS OF RADII UNDER CONFORMAL MAPS OF THE UNIT DISC

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ABSTRACT. If $E_f(R)$ is the set of endpoints of radii which have length greater than or equal to $R > 0$ under a conformal map f of the unit disc, then $\text{cap } E_f(R) = O(R^{-1/2})$ as $R \rightarrow \infty$ for the logarithmic capacity of $E_f(R)$. The exponent $-1/2$ is sharp.

Suppose \mathbf{D} is the unit disc in the plane. A well-known theorem by Beurling [Beu] states that if $f : \mathbf{D} \rightarrow \mathbf{C}$ is conformal, then the set of radii whose images under f have infinite length has vanishing logarithmic capacity. We give a quantitative version of this statement which is asymptotically sharp and improves an estimate by Pommerenke [Pom, p. 215].

Theorem. *There exists a universal constant $K > 0$ with the following property.*

Suppose $f : \mathbf{D} \rightarrow \mathbf{C}$ is a conformal map with $f'(0) = 1$. If $E_f(R)$ is the set of all $\zeta \in \partial\mathbf{D}$ with $\text{length } f([0, \zeta]) \geq R > 0$, then $\text{cap } E_f(R) \leq K/\sqrt{R}$.

On the other hand, there exist functions f , e.g. the Koebe function, for which $\text{cap } E_f(R) \geq \frac{1}{2\sqrt{R}}$ for large R .

Our theorem implies that $\text{cap } E_f(R) = O(R^{-1/2})$ as $R \rightarrow \infty$ for all conformal maps f of \mathbf{D} and that $1/2$ is the best possible constant in this statement.

1. NOTATION AND AUXILIARY RESULTS

A curve $\gamma : I \rightarrow \mathbf{C}$ is a continuous mapping of an interval $I \subseteq \mathbf{R}$. It is understood that a curve is locally rectifiable. If we speak of a curve in an open set Ω , then we allow the endpoints of the curve to lie on the boundary of the set. A curve γ in Ω connects two sets $A, B \subseteq \overline{\Omega}$, if γ has one endpoint in A and one in B . We denote by $\text{length}(\gamma) \in [0, \infty]$ the euclidean length of γ .

For the proof of our theorem we need the following version of the Gehring-Hayman theorem (cf. [GH], [Pom, p. 88]).

Theorem A. *There is a universal constant $C > 0$ with the following property. Suppose $f : \mathbf{D} \rightarrow \mathbf{C}$ is conformal, γ is a curve in \mathbf{D} with endpoints 0 and $\zeta \in \partial\mathbf{D}$, and $[0, \zeta)$ is the radius of \mathbf{D} with endpoint ζ . Then*

$$\text{length } f([0, \zeta]) \leq C \text{length}(f \circ \gamma).$$

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The modulus $\text{mod } \Gamma \in [0, \infty]$ of a family Γ of curves in an open set Ω is defined as

$$\text{mod } \Gamma = \inf_{\rho} \int_{\Omega} \rho(z)^2 dm_2(z).$$

Here m_2 is two-dimensional Lebesgue measure and the infimum is taken over all Borel measurable densities $\rho : \Omega \rightarrow [0, \infty]$ that satisfy $\int_{\gamma} \rho(z) |dz| \geq 1$ for all $\gamma \in \Gamma$, where $|dz|$ means integration with respect to euclidean arc-length. The notation for the modulus should indicate which reference set Ω we consider, but we will suppress this, since it will be clear from the context which Ω we mean. If Γ_1 is a curve family in some open set Ω_1 , $f : \Omega_1 \rightarrow \Omega_2$ is a conformal map and Γ_2 is the curve family in Ω_2 consisting of the curves $f \circ \gamma$, $\gamma \in \Gamma_1$, then $\text{mod } \Gamma_1 = \text{mod } \Gamma_2$. See [Pom, Ch. 9] for basic properties of the modulus.

We denote the logarithmic capacity of a Borel set $E \subseteq \mathbf{C}$ by $\text{cap } E$. For the definition of the logarithmic capacity see [Pom, Ch. 9].

The following statement which relates the concepts of modulus and capacity is needed in the proof of the theorem. It is part of Pfluger's theorem (cf. [Pom, p. 212]).

Theorem B. *Suppose E is a Borel subset of $\partial \mathbf{D}$ and $\Gamma_E(\epsilon)$ is the family of all curves γ in $\Omega = \{z \in \mathbf{D} : \epsilon < |z| < 1\}$ that connect $\{z \in \mathbf{D} : |z| = \epsilon\}$ and E . Then for sufficiently small $\epsilon > 0$*

$$\text{cap } E \leq \frac{1 + \epsilon}{\sqrt{\epsilon}} \exp\left(-\frac{\pi}{\text{mod } \Gamma_E(\epsilon)}\right).$$

The next lemma states a standard modulus estimate. The constant 2π in this inequality is crucial to get the right asymptotic behavior in the theorem. The usefulness of modulus estimates with sharp constants is well-known and dates back to Ahlfors's distortion theorem (cf. [Ahl]).

Lemma. *Suppose $\Omega \subseteq \mathbf{C}$ is a region and Γ is a family of curves in Ω which have one endpoint in a compact set $M \subseteq \overline{\Omega}$. Suppose M is contained in a disc of diameter $\delta > 0$ centered at the origin. If $L \geq \delta$ and $\text{length } \gamma \geq L$ for all $\gamma \in \Gamma$, then*

$$\text{mod } \Gamma \leq \frac{2\pi}{\log(1 + L/\delta)}.$$

This lemma and its proof are similar to Lem. 3.2 in [BKR].

Proof of the lemma. In addition to our assumptions on M we may assume that there exists at least one rectifiable curve in Ω which connects a point in Ω to a point in M . For otherwise it is easy to see that $\text{mod } \Gamma = 0$. (Consider test functions ρ which are equal to $\epsilon > 0$ on $B \cap \Omega$ where B is some open disc containing M and 0 elsewhere. Let ϵ tend to 0.)

For $w \in \Omega$ define $l(w) = \inf_{\gamma} \text{length}(\gamma)$, where the infimum is taken over all curves in Ω connecting w and M . The additional assumption on M implies that $l(w) < \infty$ for all $w \in \Omega$. The function l is continuous on Ω and satisfies $l(w) \geq |w| - \delta/2$ for $w \in \Omega$. Moreover, if $\gamma : [0, t_0] \rightarrow \mathbf{C}$ is a curve in Ω parameterized with respect to arc-length and if $\gamma(0) \in M$, then $l(\gamma(t)) \leq t$ for $t \in (0, t_0]$.

Define $\rho : \Omega \rightarrow [0, \infty)$ by

$$\rho(w) = \begin{cases} \frac{1}{(\log(1 + L/\delta))(\delta + l(w))} & \text{if } l(w) \leq L, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, the function ρ is Borel measurable and we claim that $\int_{\gamma} \rho(w) |dw| \geq 1$ for all $\gamma \in \Gamma$.

To see this let $\gamma \in \Gamma$ be arbitrary. We may assume that $\gamma : I \rightarrow \mathbf{C}$ has an arc-length parametrization with $I = [0, \text{length}(\gamma)]$ and that $\gamma(0) \in M$. We have $l(\gamma(s)) \leq s$ for all $s \in I \setminus \{0\}$. By assumption $\text{length}(\gamma) \geq L$ and so

$$\int_{\gamma} \rho(w) |dw| \geq \frac{1}{\log(1 + L/\delta)} \int_0^L \frac{ds}{\delta + l(\gamma(s))} \geq \frac{1}{\log(1 + L/\delta)} \int_0^L \frac{ds}{\delta + s} = 1.$$

Therefore, if $L \geq \delta$

$$\begin{aligned} \text{mod } \Gamma &\leq \int_{\Omega} \rho(w)^2 dm_2(w) \\ &= \frac{1}{[\log(1 + L/\delta)]^2} \int_{\{w \in \Omega : l(w) \leq L\}} \frac{dm_2(w)}{(\delta + l(w))^2} \\ &\leq \frac{1}{[\log(1 + L/\delta)]^2} \int_{\{w \in \mathbf{C} : |w| \leq L + \delta/2\}} \frac{dm_2(w)}{(\delta/2 + |w|)^2} \\ &= \frac{2\pi}{\log(1 + L/\delta)} + 2\pi \frac{\log 2 - 1 + \delta/(2L + 2\delta)}{[\log(1 + L/\delta)]^2} \\ &\leq \frac{2\pi}{\log(1 + L/\delta)}. \end{aligned}$$

The lemma follows. □

2. PROOF OF THE THEOREM

The idea of the proof is essentially the same as in [Pom, p. 215–216]. A limiting argument is employed in Pfluger’s theorem which is related to the concept of reduced extremal distance (cf. [Ahl]). The new ingredients in our proof are the more refined modulus estimate of the lemma and the use of the Gehring-Hayman theorem. The proof will show that for the constant K in the theorem we can take $K = \sqrt{2C}$ where C is the constant in the Gehring-Hayman theorem.

We use the notation of the theorem and may assume $f(0) = 0$. Let $\epsilon \in (0, 1)$ be arbitrary. Let $\Gamma_1(\epsilon)$ be the family of all curves in $\{z \in \mathbf{D} : \epsilon < |z| < 1\}$ connecting $\{z \in \mathbf{D} : |z| = \epsilon\}$ and $E_f(R)$. We leave it to the reader to show that the set $E_f(R)$ is a countable intersection of open subsets of $\partial\mathbf{D}$. Hence it is a Borel set.

Suppose $\gamma \in \Gamma_1(\epsilon)$ and let $z_0 \in \mathbf{D}$, $|z_0| = \epsilon$, and $\zeta \in E_f(R)$ be the endpoints of γ . Let $[0, z_0]$ be the line segment with endpoints 0 and z_0 . If we join $[0, z_0]$ and γ , then we get a curve $\tilde{\gamma}$ in \mathbf{D} connecting 0 and ζ . By the Gehring-Hayman theorem and by definition of $E_f(R)$

$$\text{length}(f \circ \tilde{\gamma}) \geq (1/C) \text{length } f([0, \zeta]) \geq R/C.$$

By Koebe’s distortion theorem (cf. [Pom, p. 9]), $|f'(z)| \leq (1 + 5\epsilon)$ if $|z| \leq \epsilon$ and $\epsilon > 0$ is sufficiently small. It follows that for small ϵ

$$\text{length}(f \circ \gamma) \geq R/C - (\epsilon + 5\epsilon^2) =: L.$$

We now apply the lemma for the region $\Omega = f(\mathbf{D} \setminus \{z \in \mathbf{D} : |z| \leq \epsilon\})$, the compact set $M = f(\{z \in \mathbf{D} : |z| = \epsilon\}) \subseteq \overline{\Omega}$ and the curve family $\Gamma_2(\epsilon) = \{f \circ \gamma : \gamma \in \Gamma_1(\epsilon)\}$. By Koebe's distortion theorem M is contained in a disc centered at the origin of diameter $\delta = 2\epsilon(1 + 3\epsilon)$ for small $\epsilon > 0$. It follows that for small $\epsilon > 0$

$$\text{mod } \Gamma_1(\epsilon) = \text{mod } \Gamma_2(\epsilon) \leq \frac{2\pi}{\log\left(\frac{R/C + \epsilon + \epsilon^2}{2\epsilon(1+3\epsilon)}\right)}.$$

Hence Pfluger's theorem implies

$$\text{cap } E_f(R) \leq \liminf_{\epsilon \rightarrow 0} \frac{(1 + \epsilon)(2 + 6\epsilon)^{1/2}}{(R/C + \epsilon + \epsilon^2)^{1/2}} = \frac{\sqrt{2C}}{\sqrt{R}}.$$

The first part of the theorem follows.

For the second part consider the Koebe function $f(z) = z/(1-z)^2$, $z \in \mathbf{C} \setminus \{1\}$. If $R > 1/4$ there exists $\phi \in (0, \pi)$ such that $R = 1/(4 \sin^2(\phi/2))$. Since $\text{length } f([0, \zeta]) \geq |f(\zeta)|$ for $\zeta \in \partial\mathbf{D}$, we have

$$A = \{e^{i\alpha} : \alpha \in [-\phi, \phi]\} \subseteq E_f(R).$$

Since the capacity of the circular arc A is $\text{cap } A = \sin(\phi/2)$ (cf. [Pom, p. 207]) we obtain $\text{cap } E_f(R) \geq \frac{1}{2\sqrt{R}}$. The theorem follows. \square

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