

LINDELÖF PROPERTY AND ABSOLUTE EMBEDDINGS

A. BELLA AND I. V. YASCHENKO

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ABSTRACT. It is proved that a Tychonoff space is Lindelöf if and only if whenever a Tychonoff space Y contains two disjoint closed copies X_1 and X_2 of X , then these copies can be separated in Y by open sets. We also show that a Tychonoff space X is weakly C -embedded (relatively normal) in every larger Tychonoff space if and only if X is either almost compact or Lindelöf (normal almost compact or Lindelöf).

1. INTRODUCTION AND RESULTS

Notations and terminology follow [En]. Unless otherwise stated, all spaces are assumed to be Tychonoff (and T_1). By βX we denote the Stone-Cech extension of the space X . Recall that the Lindelöf number of a space X , denoted by $l(X)$, is the smallest cardinal τ such that every open cover of X has a subcover of size not greater than τ . If $l(X) \leq \omega$, then X is called Lindelöf. For a non-compact space X put $\lambda(X) = \min\{|A| : A \subset X \text{ and } \text{Cl}_X A \text{ is not compact}\}$. $\lambda(X)$ might be called the index of boundedness of X , since $\lambda(X) \geq \omega_1$ if and only if X is ω -bounded.

Observe that if $\mu < \lambda(X)$, then each subset $A \subset X$ of cardinality $|A| \leq \mu$ has a complete accumulation point, so X is initially μ -compact (i.e., every open cover of X of size not bigger than μ has a finite subcover) [St, Theorem 2.2].

Two subsets A and B of a space X are said to be *completely separated* if there is a continuous function $f : X \rightarrow \mathbb{R}$ so that $f(A) = \{0\}$ and $f(B) = \{1\}$.

The main result of the paper is the following theorem, proving true the conjecture of Arhangel'skij [AT, Problem 2].

Theorem 1.1. *For a Tychonoff space X the following conditions are equivalent:*

1. X is Lindelöf.
2. *If a Tychonoff space Y contains two disjoint closed copies X_1 and X_2 of X , then these copies can be separated in Y by open sets.*

Hewitt was the first to consider the problem of whether a space X is C -embedded into every larger Tychonoff space Y (that is, every continuous function on X can be extended to a continuous function on Y). It was proved by Hewitt and Smirnov [He], [Sm] (see also [GJ, Ex. 6J]) that this is the case if and only if X is almost

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compact, i.e. $|\beta X \setminus X| \leq 1$, or, equivalently, for every two disjoint completely separated closed subsets of X at least one is compact.

Later, R.Blair proved that a space X is absolutely ν -embedded (that is, $\nu X \subset \nu Y$ whenever $X \subset Y$) if and only if X is either realcompact or almost compact [Bl]. R.Blair, A.Hager and D.Johnson proved that a space X is absolutely z -embedded (that is, each zero-set of X can be extended to a zero-set of Y whenever $X \subset Y$) if and only if X is either Lindelöf or almost compact. They also provided several interesting characterizations of such spaces [Bl], [BH], [HJ]. E.g. these are the spaces so that the $C(X)$ — ring of continuous functions on X — has no proper subalgebra which contains constant functions, separates points and closed sets, is closed under uniform convergence and is closed under inversion in $C(X)$ [HJ, Theorem 3]. Another result, which will be useful for us, says that if every pair of disjoint non-compact zero-sets of X consists of Lindelöf sets, then X itself is either Lindelöf or almost compact (and hence absolutely z -embedded) [BH, Theorem 4.1].

Recently, the following weaker version of C -embedding has attracted attention, in particular in the theory of relative topological properties.

Definition 1.2 (Arhangel'skij). A subspace X is weakly C -embedded into a space Y if every continuous real-valued function on the space X can be extended to a real-valued function on Y , which is continuous in all points of X .

Notice that, while C -embedding and C^* -embedding (if we assume that only bounded functions must have an extension) are, in general, different notions, weak C -embedding coincides with weak C^* -embedding. (Just consider an unbounded function as a function $f : X \rightarrow (-1, 1)$, extend it to the function $\tilde{f} : Y \rightarrow [-1, 1]$, which is continuous in all points of X and note that $\text{Cl}_Y f^{-1}\{-1, 1\} \cap X = \emptyset$, so we can modify \tilde{f} to have value 0 in all points y with $f(y) = \pm 1$ without any loss of continuity at the points of X .)

Every dense embedding into a Tychonoff space is a weak C -embedding [Ar], but in the case of non-dense embedding this concept is far from being trivial.

Arhangel'skij [Ar] observed that if X is almost compact or Lindelöf, then every embedding of X into a larger Tychonoff space is a weak C -embedding. Surprisingly, these are the only possibilities. Precisely, the following theorem, answering the question of Arhangel'skij [Ar, problem 15], holds.

Theorem 1.3. *A space X is weakly C -embedded into every larger Tychonoff space Y , containing X as a closed subspace, if and only if X is either almost compact, or Lindelöf.*

In [Ar] it was also asked to characterize the spaces which are (strongly) relatively normal in every larger Tychonoff space (Problems 49 and 50); see also [AT, Problems 3 and 4]. A subspace $X \subset Y$ is called (*strongly*) *relatively normal in Y* if every two disjoint closed in Y (closed in X) subsets of X can be separated by open in Y subsets (see [Ar] for a recent survey on relative topological properties).

In [AT] it was observed that all normal almost compact spaces and in [AG] that all Lindelöf spaces are strongly relatively normal in every larger Tychonoff space.

We prove here that “the converse” is true.

Theorem 1.4. *If X is a Tychonoff space, then the following conditions are equivalent:*

1. X is either normal and almost compact or Lindelöf.

2. X is strongly normal in every Tychonoff space.
3. X is normal in every Tychonoff space.
4. X is normal in every Tychonoff space, containing X as a closed subspace.
5. X is normal and every pair of non-compact disjoint closed subsets of X consists of Lindelöf sets.

2. PROOFS

Lemma 2.1. *Let P and Q be disjoint closed sets of a space X . If any Tychonoff space Y , which contains X as a closed subspace, has two disjoint open sets separating P and Q , then either both P and Q are Lindelöf, or one of them is compact.*

Proof. Take two disjoint closed subsets P and Q of X and suppose that both are non-compact and P is not Lindelöf.

Lemma 2.2. $\lambda(Q) \geq l(P) > \omega$.

Proof of Lemma 2.2. Assuming the contrary, there is $A \subset Q$ so that $\text{Cl}_Q A$ is non-compact and $|A| < l(P) = \tau$.

Since $|A| < \tau$, there is an open cover γ of the space P having no subcover of size $\leq |A|$. We may assume that γ has no subcover of size less than $|\gamma| = \tau'$. Consider $\alpha P = \text{Cl}_{\beta X} P$ — a compactification of P . Then it is easy to see that there is a $Z \subset \alpha P$ such that Z can be separated from P in αP by a $G_{\tau'}$ -set, but cannot be separated from P by any G_β -set for $\beta < \tau'$ [En, Ex. 3.12.24]. Since P is closed in X , $\alpha P \cap X = P$ and $Z \subset (\beta X \setminus X)$. Since $\text{Cl}_Q A$ is not compact and Q is closed in X , there is a point $a \in \beta X \setminus X$ so that $a \in \text{Cl}_{\beta X} A$. Consider a compactification C of the space X , obtained from βX by identifying the compact set $Z \cup \{a\}$ to a point b .

Let $S = P \cup Q \cup \{b\} \subset C$ and $T = S \times S \setminus \{(b, b)\}$. We claim that the closed subsets of T , $P \times \{b\}$ and $\{b\} \times Q$ cannot be separated in T by open sets. Indeed, let U be an open subset of T such that $\{b\} \times Q \subset U$. For every $x \in A \subset Q$ there exist open neighborhoods V_x and W_x of the points b and x in S such that $V_x \times W_x \subset U$. The set $G = \bigcap \{V_x : x \in A\}$ is a G_β set containing b with $\beta = |A| < \tau'$. Thus G meets P , say for instance in a point y . It follows that $\{y\} \times A \subset U$ and, since in S the point b is in the closure of A , we actually have $(y, b) \in \text{Cl}_T \{y\} \times A \subset \text{Cl}_T U$. Hence we cannot have $P \times \{b\} \cap \text{Cl}_T U = \emptyset$.

As a consequence, we see that X is not normal in the quotient space of $X \oplus T$, obtained by identifying P with $P \times \{b\}$ and Q with $\{b\} \times Q$. This contradiction finishes the proof of the Lemma 2.2. □

Lemma 2.3. $l(Q) \geq \lambda(Q)$, in particular Q is not Lindelöf.

Proof of Lemma 2.3. If $l(Q) = \tau < \lambda(Q)$, then the closure in Q of any set of cardinality at most τ is compact. In particular, Q is initially τ -compact and hence compact since $l(Q) = \tau$. But this contradicts the assumption that Q is not compact. □

Lemma 2.4. $\lambda(P) \geq l(Q)$.

Proof of Lemma 2.4. Use the same argument as in the proof of Lemma 2.2. □

Lemma 2.5. $\lambda(P) = l(P) = \lambda(Q) = l(Q) = \tau > \omega$ and τ is a regular cardinal.

Proof of Lemma 2.5. $\lambda(P) = l(P) = \lambda(Q) = l(Q) = \tau > \omega$ follows from Lemmas 2.2–2.4. Suppose $cf(\tau) = \mu < \tau$, i.e. $\tau = \sup\{a_\alpha : \alpha < \mu\}$. Since P is not Lindelöf, we can take an uncountable open cover γ of the space P , having no subcover of size less than $|\gamma|$. We claim that $|\gamma| = \tau$. Since $l(P) = \tau$, $|\gamma| \leq \tau$. If $|\gamma| < \tau$, then, as P is $|\gamma|$ -initially compact, γ would have a finite subcover. So, $|\gamma| = \tau$. Well order γ as $\gamma = \{U_\alpha : \alpha < \tau\}$. For every $\alpha < \mu$ let $W_\alpha = \bigcup\{U_\beta : \beta < a_\alpha\}$. Now, $\{W_\alpha : \alpha < \mu\}$ is a cover of P of size less than τ , so it has a finite subcover. Hence γ has a subcover of size less than τ —a contradiction. \square

Now we continue the proof of Lemma 2.1.

Preliminary construction. By transfinite induction we shall construct in P something like an enlarged closed copy of the ordinal τ .

Since $l(P) = \lambda(P) = \tau$, the proof of Lemma 2.5 shows that there is an open cover γ of size τ having no subcover of size less than τ . Hence by the same argument as in the proof of Lemma 2.2, there is a compact $Z \subset \alpha P = \text{Cl}_{\beta X} P$ such that Z can be separated from P in αP by a G_τ -set, but cannot be separated from P by any G_β -set for $\beta < \tau$.

By transfinite induction define for every $\alpha < \tau$ a compact subset $F_\alpha \subset P$ and an open in αP neighborhood $W_\alpha \supset Z$ in such a way that

- (1) $F_\alpha \subset \bigcap\{\text{Cl}_{\alpha P} W_\beta : \beta < \alpha\}$,
- (2) $\text{Cl}_{\alpha P} W_\alpha \cap \bigcup\{F_\beta : \beta \leq \alpha\} = \emptyset$, in particular all F_α are disjoint,
- (3) if α is a non-limit ordinal, then F_α is a one-point-set $F_\alpha = \{x_\alpha\}$,
- (4) if α is a limit ordinal, then

$$F_\alpha = \text{Cl}_P \bigcup\{F_\beta : \beta < \alpha\} \setminus \bigcup\{F_\beta : \beta < \alpha\},$$

- (5) $d(\bigcup\{F_\beta : \beta < \alpha\}) \leq \alpha$,
- (6) for every $\alpha < \tau$, $\bigcup\{F_\beta : \beta \leq \alpha\}$ is a compact subset of P .

Take an arbitrary $x \in P$ as x_0 , and let $F_0 = \{x_0\}$ and W_0 be an arbitrary neighborhood of Z so that $x_0 \notin \text{Cl}_{\alpha P} W_0$.

Let $\alpha < \tau$ and suppose that for all $\beta < \alpha$ we have F_β and W_β satisfying conditions (1)–(6). Consider two possibilities:

- I. $\alpha = \alpha' + 1$. Since Z cannot be separated from P by $G_{\alpha'}$ -sets, there is a point $x_\alpha \in P \cap \bigcap\{W_\beta : \beta \leq \alpha'\}$. Let $F_\alpha = \{x_\alpha\}$. Clearly conditions (1), (3), (5), (6) are satisfied for α . From (6) it follows that there is an open neighborhood W_α of Z so that $\text{Cl}_{\alpha P} W_\alpha \cap \bigcup\{F_\beta : \beta \leq \alpha\} = \emptyset$. Of course, even (2) is satisfied and we are done for α .
- II. α is limit ordinal. Let $F'_\alpha = \text{Cl}_P \bigcup\{F_\beta : \beta < \alpha\}$. Since (5) is true for each $\beta < \alpha$ and $\alpha < \lambda(P)$, it follows that F'_α is a compact subset of P . From (1), (2) and (6) for each $\beta < \alpha$ we have

$$F_\alpha = F'_\alpha \setminus \bigcup\{F_\beta : \beta < \alpha\} = F'_\alpha \cap \bigcap\{\text{Cl}_{\alpha P} W_\beta : \beta < \alpha\}.$$

In particular, F_α is compact. Since F'_α is a compact subset of P , there is an open neighborhood W_α of Z so that $\text{Cl}_{\alpha P} W_\alpha \cap F'_\alpha = \emptyset$. Now all conditions (1)–(6) are clearly satisfied and we are done for α .

Clearly, the W_α 's may be chosen in such a way that in addition $Z = \bigcap\{\text{Cl}_{\alpha P} W_\alpha : \alpha < \tau\}$. It then follows that $F = \bigcup\{F_\alpha : \alpha < \tau\}$ is closed in P .

The preliminary construction is now complete. We can think about F as a closed copy of τ in P where all non-limit ordinals are usual points, but all limit ordinals are enlarged to (maybe non-trivial) compact sets.

Next, apply the same method to get a similar “copy of τ ” $G = \bigcup\{G_\alpha : \alpha < \tau\}$ in Q .

Now we are ready to embed X in a space Y in such a way that the sets P and Q cannot be separated by open sets in Y . Let $T = (\tau + 1) \times (\tau + 1) \setminus \{(\tau, \tau)\}$. We shall use the well known fact that closed copies of τ in T such as the diagonal $\Delta \subset T$ and $\tau \times \{\tau\}$ cannot be separated by open sets in T (see [Wa, Lemma 2.1.12]).

The idea of our construction is to “replace” here Δ and $\tau \times \{\tau\}$ by F and G and then “glue” the resulting space to X .

Let T' be the set of all points in $\tau \times \tau \setminus \Delta$ having both coordinates non-limit. The space Y we are looking for is $X \cup T'$ equipped with the following topology:

- (1) All points of T' are isolated.
- (2) All points $x \in X \setminus (F \cup G)$ have usual neighborhood base in X .
- (3) All points $x \in \bigcup\{F_\alpha : \alpha \text{ is non-limit}\}$ have usual neighborhood base in X .
- (4) Let $x \in F_\alpha$ and α be a limit ordinal. For every open in X neighborhood W of x define a neighborhood W' of x in Y as

$$W' = W \cup \{(a, b) \in T' : F_a \subset W \text{ and } F_b \subset W\}.$$

Notice that, in the above formula, a and b are non-limit ordinals, so F_a and F_b are one-point-sets.

- (5) Let $x \in G_\alpha$ and α be a non-limit ordinal. For every open in X neighborhood W of x and every $b < \tau$ define a neighborhood W'_b of x in Y as

$$W'_b = W \cup \{(\alpha, c) \in T' : c > b\}.$$

- (6) Let $x \in G_\alpha$ and α be a limit ordinal. For every open in X neighborhood W of x and every $b < \tau$ define a neighborhood W'_b of x in Y as

$$W'_b = W \cup \{(a, c) \in T' : G_a \subset W \text{ and } c > b\}.$$

Notice that, in the above formula, a is a non-limit ordinal, so G_a is a one-point-set.

It is tedious but not difficult to realize that Y is a Tychonoff space and X is closed in Y .

We claim that F and G cannot be separated in Y by open sets. Take an arbitrary open set U of Y containing F . We shall prove that $\text{Cl}_Y U \cap G \neq \emptyset$. Take for every *limit* $\alpha < \tau$ a point $x_\alpha \in F_\alpha$ and basic neighborhood $W'(x_\alpha) \subset U$. Observe that, by definition of F_α , the set $B_\alpha = \{b < \alpha : b \text{ is non-limit and } F_b \subset W\}$ is unbounded in α . For every α pick an arbitrary $f(\alpha) \in B_\alpha$. Since for each α $f(\alpha) < \alpha$, we can apply the Pressing-Down Lemma [Ku, Theorem 2.3] to find a non-limit $b < \tau$ so that $b = f(\alpha)$ for unboundedly (in τ) many α . We claim that for every $c < \tau$ there are $d > c$ and $\alpha \in f^{-1}(b)$ so that $(b, d) \in W'(x_\alpha)$. Indeed, there is $\alpha \in f^{-1}(b)$ with $\alpha > c$. Since B_α is unbounded in α , there is $d \in B_\alpha$ with $d > c$. Now both $b, d \in B_\alpha$, hence $(b, d) \in W'(x_\alpha)$ and our claim holds. Next, consider G_b and recall that, since b is a non-limit ordinal, G_b is a one-point-set, $G_b = \{g\}$. From the last claim it immediately follows that $g \in \text{Cl}_Y U$, therefore $\text{Cl}_Y U \cap G \neq \emptyset$ and the proof of Lemma 2.1 is complete. \square

Lemma 2.6. *If X is weakly C -embedded in every larger Tychonoff space Y , containing X as a closed subspace, then for every two disjoint completely separated closed subsets P and Q of X either both are Lindelöf or one of them is compact.*

Proof. Let $f : X \rightarrow \mathbb{R}$ be a continuous function such that $f(P) = \{0\}$ and $f(Q) = \{1\}$. Suppose that X is a closed subspace of a space Y . By our assumption, there is a function $\tilde{f} : Y \rightarrow \mathbb{R}$, which is continuous at every point of X and satisfies $\tilde{f}|_X = f$. Consequently, P and Q are separated by open sets in Y and hence, by Lemma 2.1, either both P and Q are Lindelöf or one of them is compact. \square

Proof of Theorem 1.3. Sufficiency — the only part we need to prove — follows from Lemma 2.6 and the result of Blair, Hager and Johnson cited in the introduction. \square

Proof of Theorem 1.4. $1 \Rightarrow 2$ was observed in [AT]. $5 \Rightarrow 1$ follows from the result of Blair, Hager and Johnson, cited in the introduction. $2 \Rightarrow 3 \Rightarrow 4$ is evident. $4 \Rightarrow 5$ follows from Lemma 2.1 and the fact that X is normal, being relatively normal in itself. \square

Proof of Theorem 1.1. $1 \Rightarrow 2$ is easy. $2 \Rightarrow 1$ follows from Lemma 2.1 for $P = X_1$, $Q = X_2$ and $X \oplus X$ “as” X . \square

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REFERENCES

- [Ar] A.V.Arangel'skij, *Relative topological properties and relative topological spaces*, Topology and Appl. **70** (1996), 87-89. MR **97f**:54030
- [AG] A.V.Arangel'skij and H.M.M.Genedi, *Beginnings of the theory of relative topological properties*, General Topology, Spaces and Mappings, MGU, Moscow, 1989, pp. 87-89.
- [AT] A. V. Arangel'skij and J.Tartir, *A characterization of compactness by a relative separation property*, Q and A in Topology **14** (1996), 49–52. MR **96m**:54037
- [BH] R.L.Blair and A.W.Hager, *Extensions of Zero-sets and of Real-valued Functions*, Math. Z. **136** (1974), 41-52. MR **52**:6652
- [Bl] R.L.Blair, *On ν -embedded sets in topological spaces*, Proceedings of the Second Pittsburgh Symposium on General Topology (Pittsburgh 1972), Springer-Verlag, Berlin, 1974. MR **50**:11136
- [HJ] A.W.Hager and D.G.Johnson, *A note on certain subalgebras of $C(X)$* , Canadian J. Math. **20** (1968), 389–393. MR **36**:5697
- [En] R.Engelking, *General Topology*, PWN, Warszawa, 1977. MR **58**:18316b
- [GJ] L.Gillman and M.Jerison, *Rings of continuous functions*, Springer-Verlag, Berlin, 1976. MR **53**:11352
- [He] E.Hewitt, *A note on extensions of continuous functions*, An. Acad. Brasil. Ci. **21** (1949), 175–179. MR **11**:194c
- [Ku] K. Kunen, *Combinatorics*, Handbook of Mathematical Logic (J. Barwise, eds.), Elsevier S.P., North-Holland, Amsterdam, 1977.
- [Sm] Yu.Smirnov, *Mappings of systems of open sets (in Russian)*, Matem. Sbornik **31** (1952), 152–166. MR **14**:303g
- [St] R. M. Stephenson, Jr, *Initially k -compact and related spaces*, Handbook of Set-Theoretic Topology (K.Kunen and J. Vaughan, eds.), Elsevier S.P., North-Holland, Amsterdam, 1984, pp. 603–632. MR **86i**:54024

- [Wa] S. Watson, *The Construction of Topological Spaces: Planks and Resolutions*, Recent Progress in General Topology (M. Husek and J. van Mill, eds.), Elsevier S.P., North-Holland, Amsterdam, 1992, pp. 673–757. CMP 93:15

DIPARTIMENTO DI MATEMATICA, CITTA UNIVERSITARIA, VIALE A.DORIA 6, 95125, CATANIA,
ITALY

E-mail address: `bella@dipmat.unict.it`

MOSCOW CENTER FOR CONTINUOUS MATHEMATICAL EDUCATION, B.VLAS'EVSKIJ PER. 11,
121002, MOSCOW, RUSSIA

E-mail address: `ivan@mcme.ru`