EQUILIBRIUM PROBLEMS ASSOCIATED
WITH FAST DECREASING POLYNOMIALS

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ABSTRACT. The determination of the support of the equilibrium measure in
the presence of an external field is important in the theory of weighted polyno-
mials on the real line. Here we present a general condition guaranteeing that
the support consists of at most two intervals. Applying this to the external
fields associated with fast decreasing polynomials, we extend previous results
of Totik and Kuijlaars-Van Assche. In the proof we use the iterated balayage
algorithm which was first studied by Dragnev.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $w(x) = \exp(-Q(x))$ be a positive continuous weight defined on a compact
set $\Sigma \subset \mathbb{R}$. The study of weighted polynomials $w^n P_n$, $\deg P_n \leq n$, has found
important applications in the theory of orthogonal polynomials on the real line; see
e.g. [4], [10], [12], [17]. See also [14] where other applications can be found as well.

In this theory a fundamental role is played by a probability measure μ_w with
compact support $S_w \subset \Sigma$ characterized by the equations

$$U^{\mu_w}(x) + Q(x) = F_w, \qquad \text{q.e. on } S_w,$$
(1.1)
$$U^{\mu_w}(x) + Q(x) \geq F_w, \qquad \text{q.e. on } \Sigma.$$

Here F_w is a uniquely determined constant, U^{μ_w} denotes the logarithmic potential
of μ_w, i.e.,

$$U^{\mu_w}(z) = \int \log \frac{1}{|z-t|} d\mu_w(t), \qquad z \in \mathbb{C},$$

and q.e. means quasi-everywhere, i.e., except for a set of logarithmic capacity zero.
The measure μ_w is called the equilibrium measure (or extremal measure) associated
with the external field Q. Its support S_w is called the extremal support. See [8] for
an interpretation of (1.1) as a contact problem in elasticity theory.

A complicating factor in solving (1.1) is that the support S_w is not known in
advance. It is part of the problem to determine the nature of S_w. The simplest

case is that S_w is an interval, since then there are explicit formulas available for the measure μ_w. This holds for example, if Σ is an interval and the external field Q is convex, which is the case that is studied most in the literature.

Recently, external fields of the form

$$(1.2) \qquad\qquad Q(x) = -cx^\alpha, \qquad x \in [0,1],$$

were found to be of interest in connection with the construction of fast decreasing polynomials. In this context, a sequence of polynomials $\{P_n\}_{n=0}^\infty$, $\deg P_n \le n$, is fast decreasing if for some $M > 0$, we have

$$(1.3) \qquad\qquad P_n(0) = 1, \qquad |P_n(x)| \le M \exp(-cnx^\alpha), \quad x \in [0,1].$$

The right-hand side gives a prescribed rate of decrease of the polynomials on the interval $[0,1]$. The existence of polynomials satisfying (1.3) depending on the parameters α and c was investigated in a number of papers [6], [7], [16]. The results may be summarized as follows. Let $w(x) = \exp(cx^\alpha)$ on $[0,1]$ and let μ_w be the equilibrium measure associated with the external field (1.2). Then fast decreasing polynomials (1.3) exist if and only if equality holds in (1.1) for $x = 0$, i.e., $U^{\mu_w}(0) = F_w$. In particular, a sufficient condition is that $0 \in S_w$. Therefore, the determination of the support of μ_w is of importance here.

The case $\alpha \le 1$ was studied by Lubinsky and Totik [11] and Totik [16], [17], who found that polynomials satisfying (1.3) exist if and only if

$$(1.4) \qquad\qquad c \le c_0(\alpha) := \frac{2\sqrt{\pi}\Gamma(\alpha)}{\Gamma(\alpha - 1/2)}, \qquad 1/2 < \alpha \le 1,$$

while for $\alpha \le 1/2$, fast decreasing polynomials satisfying (1.3) do not exist if $c > 0$. For $\alpha \le 1$, the external field is convex and the extremal support is an interval containing 1 for every c. Thus the problem comes down to determining whether the support is $[0,1]$.

The case $\alpha > 1$ is more complicated, since Q is no longer convex and the extremal supports are not necessarily intervals. This case was studied by Kuijlaars and Van Assche [7], who determined the optimal constant $c_0(\alpha)$, thereby solving a problem posed in [18]. It is not given by (1.4) but by a more complicated formula; see [7] for details. What was proved is that for $c \ge c_0(\alpha)$, the extremal support is an interval of the form $[b,1]$ with $b > 0$. For $c = c_0(\alpha)$, the support does not contain 0, but still the equality $U^{\mu_w}(0) = F_w$ holds.

What remained open is what the extremal support looks like for $c < c_0(\alpha)$ and it is one of the purposes of this paper to fill this gap. We prove that the extremal support can consist of at most two intervals.

Theorem 1. *Let $\alpha > 1$ and $c > 0$. Let $w(x) = \exp(-Q(x))$ with Q given by (1.2). Then there are positive constants $c_1(\alpha) < c_0(\alpha)$, depending only on α, such that the following hold.*

(1) *If $c \le c_1(\alpha)$, then $S_w = [0,1]$.*
(2) *If $c \in (c_1(\alpha), c_0(\alpha))$, then S_w has the form $[0,a] \cup [b,1]$ with $0 < a < b < 1$.*
(3) *If $c \ge c_0(\alpha)$, then S_w has the form $[b,1]$ with $0 < b < 1$.*

That the support cannot be worse than two intervals may come somewhat as a surprise, since numerical calculations of weighted Leja points seemed to indicate that several intervals could appear and it was even suggested that for some α, chaos might appear [18]. Knowing the above result one has to conclude that the

numerical calculations were not reliable. The calculation of a large number of Leja points seems to be a process which is very unstable.

Theorem 1 will follow from a result which gives general conditions guaranteeing that the extremal support consists of at most two intervals. To describe these conditions, we assume that $\Sigma = [a, b]$ and that Q is differentiable with a Hölder continuous derivative, i.e., $Q \in C^{1+\epsilon}([a, b])$ for some $\epsilon > 0$. Then it is well-known [3], [13] that the singular integral equation

$$(1.5) \qquad \int_a^b \frac{v(t)}{x - t} \, dt = Q'(x), \qquad a < x < b,$$

has a unique solution satisfying

$$(1.6) \qquad \int_a^b v(t) \, dt = 1.$$

Note that $v(t)$ is not necessarily ≥ 0. But if it is, then it follows that $v(t) \, dt$ is the equilibrium measure with external field Q. The solution $v(t)$ is given by the explicit expression

$$(1.7) \quad v(t) = \frac{1}{\pi \sqrt{(b - t)(t - a)}} \left[1 + \frac{1}{\pi} \int_a^b \frac{Q'(s)}{s - t} \sqrt{(b - s)(s - a)} \, ds \right], \quad a < t < b,$$

where the integral is again a Cauchy principal value integral, see [3, p. 428].

Then we have the following result.

Theorem 2. *Suppose $w = \exp(-Q)$ with $Q \in C^{1+\epsilon}([a, b])$ for some $\epsilon > 0$ and let $v(t)$ be given by (1.7).*

(1) *If $\pi \sqrt{(b - t)(t - a)} \, v(t)$ is increasing on $[a, b]$, then the extremal support S_w is an interval containing b.*

(2) *If $\pi \sqrt{(b - t)(t - a)} \, v(t)$ is decreasing on $[a, b]$, then the extremal support is an interval containing a.*

(3) *If there is a $t_0 \in (a, b)$ such that $\pi \sqrt{(b - t)(t - a)} \, v(t)$ is decreasing on $[a, t_0]$ and increasing on $[t_0, b]$, then the extremal support is either an interval containing a, or an interval containing b, or the union of an interval containing a with an interval containing b.*

We note that also parts (1) and (2) have not appeared in the literature before.

The proof of Theorem 2 is based on an algorithmic approach to solve the equilibrium problem (1.1). This so-called iterated balayage algorithm was first studied by Dragnev [1] in his thesis. It is described in Section 2 and the properties that are needed for Theorem 2 are proved. In Section 3 we give the proof of Theorem 2 and in Section 4 we prove Theorem 1 by showing that for the external field Q from (1.2), the condition of Theorem 2(1) is satisfied.

Throughout the paper we use σ^+ and σ^- to denote the positive and negative parts, respectively, in the Jordan decomposition of a signed measure $\sigma = \sigma^+ - \sigma^-$.

2. THE ITERATED BALAYAGE ALGORITHM

We start with a lemma.

Lemma 3. *Suppose $w(x) = \exp(-Q(x))$, $x \in \Sigma$, and*

$$(2.1) \qquad Q(x) + U^\sigma(x) = C, \qquad for \ q.e. \ x \in \Sigma$$

where C is a constant and σ is a signed measure on Σ such that $\int d\sigma = 1$. Suppose U^{σ^+} is continuous and U^{σ^-} is bounded on Σ. Then

$$(2.2) \qquad \mu_w \leq \sigma^+ \qquad and \qquad S_w \subset \mathrm{supp}(\sigma^+).$$

Proof. From (1.1) and (2.1) it follows that

$$U^{\sigma^+}(x) = U^{\mu_w + \sigma^-}(x) + C - F_w, \qquad \text{q.e. on } S_w,$$

$$(2.3) \qquad U^{\sigma^+}(x) \leq U^{\mu_w + \sigma^-}(x) + C - F_w, \qquad \text{q.e. on } \Sigma.$$

Since $\int d\sigma = 1$, we have $\|\sigma^+\| - \|\sigma^-\| = 1$, so that $\|\sigma^+\| = \|\mu_w + \sigma^-\|$. Therefore we can apply the principle of domination for logarithmic potentials (see e.g. [14], [15]), and we obtain that the inequality (2.3) holds for every $x \in \mathbb{C}$. Then by a theorem of de La Vallée Poussin (see [14] or [17]), we have $(\mu_w + \sigma^-)|_{S_w} \leq (\sigma^+)|_{S_w}$. Hence $\mu_w \leq \sigma^+$ and the lemma follows. $\qquad \square$

The lemma is closely related to Theorem 2.6 of [2].

Lemma 3 provides the basis for the iterated balayage algorithm. We recall the notion of balayage onto a compact set; see [9], [14], [15], [17]. If K is a compact subset of the real line with positive logarithmic capacity and ν is any finite positive measure on $\mathbb{C}$ with compact support, then there is a unique positive measure $\hat{\nu}$ supported on K such that $\|\hat{\nu}\| = \|\nu\|$, $\hat{\nu}$ vanishes on the irregular points of K and for some constant C,

$$(2.4) \qquad U^{\hat{\nu}}(z) = U^{\nu}(z) + C, \qquad \text{q.e. on } K.$$

The measure $\hat{\nu}$ is called the balayage of ν onto K and we denote it by $Bal(\nu; K)$. For a signed measure ν, we define $Bal(\nu; K) := Bal(\nu^+; K) - Bal(\nu^-; K)$.

We introduce an operator J defined on a subset of the space of finite signed measures with compact support in $\mathbb{R}$. The domain of J consists of all signed measures σ with $\int d\sigma = 1$ having the property that $\mathrm{cap}(\mathrm{supp}(\sigma^+)) > 0$. For such σ, we define

$$(2.5) \qquad J(\sigma) := \sigma^+ - Bal(\sigma^-; \mathrm{supp}(\sigma^+)) = Bal(\sigma; \mathrm{supp}(\sigma^+)).$$

Thus the operator J sweeps the negative part of the measure σ onto the support of the positive part, so that in particular $J(\sigma)^+ \leq \sigma^+$. For each $k \geq 1$, $J^k(\sigma)$ denotes the kth iterate of σ under J, if it is defined.

Suppose we are in the situation of Lemma 3. Then it follows from (2.2) that $\mathrm{cap}(\mathrm{supp}(\sigma^+)) \geq \mathrm{cap}(S_w) > 0$, so that $\sigma_1 := J(\sigma)$ is defined. Put $\Sigma_1 := \mathrm{supp}(\sigma^+)$. Then we have by (2.1) and the property (2.4) of balayage that

$$Q(x) + U^{\sigma_1}(x) = C_1, \qquad \text{for q.e. } x \in \Sigma_1.$$

It can be shown that Lemma 3 applies again so that $\mu_w \leq \sigma_1^+ \leq \sigma^+$. Continuing in this way, we obtain that $\sigma_k := J^k(\sigma)$ exists for every k, and that

$$(2.6) \qquad S_w \subset \mathrm{supp}(\sigma_k^+) \quad \text{and} \quad \sigma_k^+ \geq \sigma_{k+1}^+ \geq \cdots \geq \mu_w.$$

It seems reasonable to expect from (2.6) that the sequence $\{\sigma_k^+\}$ converges to μ_w and indeed, this can be proved under quite general assumptions (see [1]). It is not known if the algorithm converges to μ_w in the full generality presented here. However, for the proof of Theorem 2 we need it only in a very special case where the convergence can be easily shown. This will be done in the next section.

3. Proof of Theorem 2

We need one more lemma about balayage.

Lemma 4. *Let K be a compact subset of $\mathbb{R}$. Let ν be a positive measure with compact support with $\nu(K) = 0$, and let $\hat{\nu}$ be the balayage of ν onto K.*

(1) *Suppose $K = [a, b]$. Then $\hat{\nu}$ is absolutely continuous with respect to Lebesgue measure. If $\operatorname{supp}(\nu) \subset (-\infty, a]$, then $\pi\sqrt{(b-t)(t-a)}\,d\hat{\nu}/dt$ is decreasing on $[a, b]$. If $\operatorname{supp}(\nu) \subset [b, \infty)$, then $\pi\sqrt{(b-t)(t-a)}\,d\hat{\nu}/dt$ is increasing on $[a, b]$.*

(2) *Suppose $K = [a, b] \cup [c, d]$ with $a < b < c < d$. If $\operatorname{supp}(\nu) \subset [b, c]$, then $\hat{\nu}$ is absolutely continuous and $\pi\sqrt{(d-t)(t-a)}\,d\hat{\nu}/dt$ is increasing on $[a, b]$ and decreasing on $[c, d]$.*

Proof. (1) This is immediate from the explicit representation

$$\pi\sqrt{(b-t)(t-a)}\,\frac{d\hat{\nu}}{dt} = \int \frac{\sqrt{(b-s)(a-s)}}{|s-t|}\,d\nu(s), \qquad a < t < b;$$

see e.g. [14, Section II.4].

(2) It is enough to prove (2) for $\nu = \delta_s$, the point mass of a point s where $b < s < c$, since for general ν we have

$$\frac{d\hat{\nu}}{dt} = \int_b^c \frac{d\hat{\delta}_s}{dt}\,d\nu(s).$$

Without loss of generality we may further assume that $s = 0$. We note that the usual equilibrium measure (with external field $Q \equiv 0$) of the set $[1/b, 1/a] \cup [1/d, 1/c]$ has a density given by

$$u(t) := \frac{|t - X|}{\pi\sqrt{|t - 1/a||t - 1/b||t - 1/c||t - 1/d|}}, \qquad t \in [1/b, 1/a] \cup [1/d, 1/c],$$

with X some point in the interval $(1/a, 1/d)$; see [15, Lemma 4.4.1]. This means that for some constant C_1,

$$-\int \log|x - t|u(t)dt = C_1, \qquad x \in [1/b, 1/a] \cup [1/d, 1/c].$$

If we change variables $t \mapsto 1/t$ and write $1/x$ instead of x, we obtain after simple calculations

$$-\int \log|x - t|\frac{u(1/t)}{t^2}dt = -\log|x| + C_2, \qquad x \in [a, b] \cup [c, d],$$

with a (different) constant C_2. Thus

$$\frac{d\hat{\delta}_0}{dt} = \frac{u(1/t)}{t^2} = \frac{|1 - tX|}{|t|}\frac{\sqrt{abcd}}{\pi\sqrt{|t - a||t - b||t - c||t - d|}}, \qquad t \in [a, b] \cup [c, d].$$

Then (2) easily follows, since $X \in (1/a, 1/d)$. $\qquad\qquad\square$

Lemma 5. (1) *Let $a < b$ and $v \in C((a, b))$ such that $\int_a^b v(t)\,dt = 1$. Put $d\sigma(t) = v(t)\,dt$. Suppose $\pi\sqrt{(b-t)(t-a)}\,v(t)$ is increasing on $[a, b]$. Then $\operatorname{supp}(\sigma^+) = [a^*, b]$ for some $a^* \in [a, b)$, and $J(\sigma)$ exists and has density v^* on $[a^*, b]$ such that $\pi\sqrt{(b-t)(t-a^*)}\,v^*(t)$ is increasing on $[a^*, b]$.*

(2) *Let $a < b$ and $v \in C((a,b))$ such that $\int_a^b v(t)\,dt = 1$. Put $d\sigma(t) = v(t)\,dt$. Suppose $\pi\sqrt{(b-t)(t-a)}\,v(t)$ is decreasing on $[a,b]$. Then $\operatorname{supp}(\sigma^+) = [a,b^*]$ for some $b^* \in (a,b]$, and $J(\sigma)$ exists and has density v^* on $[a,b^*]$ such that $\pi\sqrt{(b^*-t)(t-a)}\,v^*(t)$ is increasing on $[a,b^*]$.*

(3) *Let $a < b \le c < d$ and $v \in C((a,b) \cup (c,d))$ with $\int_a^b v(t)\,dt + \int_c^d v(t)\,dt = 1$. Put $d\sigma(t) = v(t)\,dt$. Suppose $\pi\sqrt{(d-t)(t-a)}\,v(t)$ is decreasing on $[a,b]$ and increasing on $[c,d]$. Then there are 3 possibilities:*

  (a) $\operatorname{supp}(\sigma^+) = [a,b^*]$ *for some $b^* \in (a,b]$. Then $J(\sigma)$ exists and has a continuous density v^* on (a,b^*) such that $\pi\sqrt{(b^*-t)(t-a)}\,v^*(t)$ is decreasing on $[a,b^*]$.*

  (b) $\operatorname{supp}(\sigma^+) = [c^*,d]$ *for some $c^* \in [c,d)$. Then $J(\sigma)$ exists and has a continuous density v^* on (c^*,d) such that $\pi\sqrt{(d-t)(t-c^*)}\,v^*(t)$ is increasing on $[c^*,d]$.*

  (c) $\operatorname{supp}(\sigma^+) = [a,b^*] \cup [c^*,d]$ *for some $b^* \in (a,b]$ and $c^* \in [c,d)$. Then $J(\sigma)$ exists and has a continuous density v^* on $(a,b^*) \cup (c^*,d)$ such that $\pi\sqrt{(d-t)(t-a)}\,v^*(t)$ is decreasing on $[a,b^*]$ and increasing on $[c^*,d]$.*

Proof. (1) Since $\int d\sigma = 1$, it is clear that $\operatorname{supp}(\sigma^+)$ has the form $[a^*,b]$ with $a^* < b$. Thus $\operatorname{cap}(\operatorname{supp}(\sigma^+)) > 0$, so that $J(\sigma)$ exists. This measure has a continuous density

$$v^*(t) = v(t) - \frac{d\hat{\sigma}^-}{dt}(t), \qquad t \in (a^*,b),$$

where $\hat{\sigma}^-$ is the balayage of σ^- onto $\operatorname{supp}(\sigma^+)$. From Lemma 4(a) we see that $\pi\sqrt{(b-t)(t-a^*)}\,d\hat{\sigma}^-/dt$ decreases on $[a^*,b]$, while we get from the assumptions that $\pi\sqrt{(b-t)(t-a^*)}\,v(t)$ increases on $[a^*,b]$. This proves (1).

(2) The proof of part (2) is analogous.

(3) It is clear that $\operatorname{supp}(\sigma^+)$ has one of the forms (a), (b) or (c) and that in all cases $J(\sigma)$ exists and has a continuous density

$$v^*(t) = v(t) - \frac{d\hat{\sigma}^-}{dt}(t), \qquad t \in \operatorname{int}(\operatorname{supp}(\sigma^+)),$$

where $\hat{\sigma}^-$ is the balayage of σ^- onto $\operatorname{supp}(\sigma^+)$. Using Lemma 4(1) in cases (a) and (b) and Lemma 4(2) in case (c), we obtain (3) in the same way as we obtained part (1). □

Having Lemma 5 we are ready for the proof of Theorem 2.

Proof of Theorem 2. (1) Let $d\sigma_0(t) := v(t)\,dt$ and put $\sigma_k := J^k(\sigma_0)$, $k \ge 1$. Using Lemma 5(1) repeatedly, we see that these measures exist. It is also clear from Lemma 5(1) that for some non-decreasing sequence $\{a_k\}_{k=0}^\infty$, we have $\operatorname{supp}(\sigma_{k-1}^+) = [a_k,b]$, so that σ_k is the balayage of σ_0 onto $[a_k,b]$, and that $\pi\sqrt{(b-t)(t-a_k)}v_k(t)$ increases on $[a_k,b]$, where v_k is the density of σ_k. Now let

$$a_\infty := \lim_{k \to \infty} a_k.$$

Since $\mu_w \le \sigma_k^+$ for every k, see (2.6), it follows that $\operatorname{supp}(\mu_w) \subset [a_\infty,b]$ and therefore $a_\infty < b$. Let σ_∞ denote the balayage of σ_0 onto $[a_\infty,b]$. Since σ_k is the balayage onto $[a_k,b]$, we have

(3.1) $$\lim_{k \to \infty} \sigma_k = \sigma_\infty$$

where the convergence is in weak* sense. Since $\text{supp}(\sigma_k^-) = [a_k, a_{k+1}]$ and the total masses $\|\sigma_k^-\|$ are decreasing (because of (2.6) and the fact that $\|\sigma_k^+\| = 1 + \|\sigma_k^-\|$), it follows that

$$(3.2) \qquad \lim_{k \to \infty} \sigma_k^- = c\delta_{a_\infty}$$

with $c \geq 0$. Next we note that $v_{k+1} \leq v_k \leq v$ on $[a_{k+1}, b]$. This readily implies that

$$(3.3) \qquad \lim_{k \to \infty} \sigma_k^+ = v_\infty(t)\, dt, \qquad a_\infty < t < b,$$

with $v_\infty(t) := \lim_k v_k(t)$, $a_\infty < t < b$. Then by (3.1), (3.2), (3.3), we have that $\sigma_\infty = v_\infty(t)\, dt - c\delta_{a_\infty}$. This implies that $c = 0$ since σ_∞ has no point masses, being the balayage onto $[a_\infty, b]$. Consequently, σ_∞ is a non-negative measure and it follows that $\sigma_\infty = \mu_w$. Since $\pi\sqrt{(b - t)(t - a_\infty)}\, v_\infty(t)$ is increasing on $[a_\infty, b]$, it follows that S_w is an interval containing b. It could be strictly smaller than $[a_\infty, b]$ if v_∞ vanishes on some interval containing a_∞. This proves part (1) of the theorem.

(2) The proof of (2) is analogous to the proof of part (1).

(3) We closely follow the proof of part (1). We put $d\sigma_0(t) := v(t)\, dt$ and $\sigma_k := J^k(\sigma_0)$, $k \geq 1$. Also put $a_0 = a$, $b_0 = c_0 = t_0$ and $d_0 = b$.

Using Lemma 5(3)(c) repeatedly, we find that there is an $N \geq 0$, a non-increasing sequence $\{b_k\}_{k=0}^N$ and a non-decreasing sequence $\{c_k\}_{k=0}^N$ such that

$$\text{supp}(\sigma_{k-1}^+) = [a_0, b_k] \cup [c_k, d_0], \qquad k \geq 1,$$

and $\pi\sqrt{(d_0 - t)(t - a_0)}\, d\sigma_k/dt(t)$ decreases on $[a_0, b_k]$ and increases on $[c_k, d_0]$. The number N may be finite or infinite, depending on whether Lemma 5(3)(c) applies a finite or infinite number of times. If it is finite, then we have that cases (a) or (b) of Lemma 5(3) apply to σ_N and we are back to parts (1) or (2). It then follows from what has been proven there that S_w is an interval containing a or an interval containing b.

If N is infinite, we put

$$b_\infty := \lim_{k \to \infty} b_k, \qquad c_\infty := \lim_{k \to \infty} c_k$$

and we have that σ_k is the balayage of σ_0 onto $[a_0, b_k] \cup [c_k, d_0]$. Then

$$\lim_{k \to \infty} \sigma_k = \sigma_\infty$$

where σ_∞ is the balayage of σ_0 onto $[a_0, b_\infty] \cup [c_\infty, d_0]$. Now, just as in the proof of part (1) it follows that $\mu_w = \sigma_\infty$ with a density v_∞ given by

$$v_\infty(t) := \lim_{k \to \infty} \frac{d\sigma_k}{dt}(t), \qquad t \in (a_0, b_\infty) \cup (c_\infty, d_0).$$

Since $\pi\sqrt{(d_0 - t)(t - a_0)}\, v_\infty(t)$ decreases on $[a_0, b_\infty]$ and increases on $[c_\infty, d_0]$, it follows that S_w is either an interval containing a_0 or an interval containing d_0 or consists of two intervals. It is, e.g., an interval containing a_0 if $c_\infty = d_0$ or if v_∞ vanishes on (c_∞, d_0).

This completes the proof of Theorem 2. $\square$

4. Proof of Theorem 1

For $\alpha > 1$, we define

$$(4.1) \qquad G_\alpha(t) := -\frac{1}{\pi} \int_0^1 \frac{s^{\alpha-1}}{s - t} \sqrt{s(1 - s)}\, ds, \qquad 0 \leq t \leq 1,$$

where the integral is a principal value integral if $0 < t < 1$. For the proof of Theorem 1 we need the following property of the function G_α.

Lemma 6. *Let $\alpha > 1$. There is a $t_0 \in (0,1)$ such that G_α decreases on $[0, t_0]$ and increases on $[t_0, 1]$.*

Proof. Put $k := [\alpha] + 1$, where $[\alpha]$ denotes the integer part of α. The lemma follows from the following three properties of G_α.

(1) $G_\alpha^{(l)}(0) < 0$ for all $l = 0, 1, \ldots, k-1$.
(2) $G_\alpha(1) > 0$.
(3) $G_\alpha^{(k)}(t) \geq 0$ for $0 < t < 1$.

Indeed, it follows from (3) that $G_\alpha^{(k-1)}$ is increasing on $[0, 1]$ and therefore it has at most one zero. If there were no zero, then we first get from (1) that $G_\alpha^{(k-1)} < 0$ on $(0, 1)$ and then it would follow from (1) that $G_\alpha < 0$ on $(0, 1)$, which is impossible because of (2). Thus $G_\alpha^{(k-1)}$ has exactly one zero and so $G_\alpha^{(k-2)}$ decreases until it reaches a minimum and then increases. By (1) with $l = k - 2$, $G_\alpha^{(k-2)}$ has at most one zero and in the same way as before it has exactly one zero. Continuing in this way, we obtain that G'_α has exactly one zero $t_0 \in (0, 1)$ and the lemma follows.

So what remains is to prove (1), (2) and (3).

(1) For $l \leq k - 1$, it is easy to obtain from (4.1) that

$$G_\alpha^{(l)}(0) = -\frac{l!}{\pi} \int_0^1 s^{\alpha - l - 3/2}(1 - s)^{1/2}\, ds < 0,$$

which is (1). Here it may be noted that $G_\alpha^{(l)}(0) = -\infty$ if $l \geq \alpha - 1/2$.

(2) follows immediately from (4.1).

(3) Put $\phi(x) := x^{\alpha+1} - x^\alpha$. From (4.1) we note that

$$(4.2) \qquad G_\alpha(t) = \frac{1}{\pi}\int_0^1 \frac{\phi(s)}{s - t}\frac{ds}{\sqrt{s(1-s)}} = \frac{1}{\pi}\int_0^1 \frac{\phi(t) - \phi(s)}{t - s}\frac{ds}{\sqrt{s(1-s)}}.$$

The Taylor expansion of ϕ with remainder term gives

$$\frac{\phi(t) - \phi(s)}{t - s} = \sum_{j=1}^k \frac{1}{j!}(t - s)^{j-1}\phi^{(j)}(s) + \frac{1}{k!}\int_s^t \frac{(t - x)^k}{t - s}\phi^{(k+1)}(x)\, dx.$$

The kth derivative of the polynomial part vanishes. Thus

$$\left(\frac{d}{dt}\right)^k \left(\frac{\phi(t) - \phi(s)}{t - s}\right) = \frac{1}{k!}\int_s^t \left(\frac{d}{dt}\right)^k \left(\frac{(t - x)^k}{t - s}\right)\phi^{(k+1)}(x)\, dx$$

$$(4.3) \qquad\qquad\qquad = \int_s^t \frac{(x - s)^k}{(t - s)^{k+1}}\phi^{(k+1)}(x)\, dx.$$

Observe that $\phi^{(k+1)}(x) \geq 0$ by the choice of k. Thus (4.3) is non-negative for all $s, t \in (0, 1)$ and then (3) follows from this and (4.2). $\square$

Remark. The function G_α can be expressed in terms of the Gauss hypergeometric function as

$$G_\alpha(t) = \frac{\Gamma(\alpha + 1/2)}{\sqrt{\pi}\Gamma(\alpha + 1)}F(-\alpha, 1; 1/2; 1 - t);$$

see formula 3.228.3 in [5]. The proof of Lemma 6 can also be based on this formula. We thank Walter Van Assche for this remark.

Proof of Theorem 1. Let $v(t)$, $0 < t < 1$, be given by (1.7) with $a = 0$ and $b = 1$, so that

$$(4.4) \qquad v(t) = \frac{1}{\pi\sqrt{t(1-t)}}\left[1 + \alpha c G_\alpha(t)\right], \qquad 0 < t < 1,$$

with G_α given by (4.1). Then by Lemma 6 and (4.4), $\pi\sqrt{t(1-t)}\,v(t)$ is decreasing on $[0, t_0]$ and increasing on $[t_0, 1]$. It follows from Theorem 2(3) that S_w is either an interval containing 0, or an interval containing 1, or the union of an interval containing 0 and an interval containing 1.

If v is non-negative on $[0, 1]$, then $S_w = [0, 1]$ and this is the case if and only if $c \leq c_1(\alpha)$ with

$$c_1(\alpha) := -\frac{1}{\alpha G_\alpha(t_0)}$$

which is a positive number. Hence we have (1).

It was proved in [7, Theorem 4] that there exists a constant $c_0(\alpha)$ such that S_w is an interval containing 1 for $c \geq c_0(\alpha)$ and $0 \in S_w$ for $c < c_0(\alpha)$. This gives (3).

In the remaining case $c \in (c_1(\alpha), c_0(\alpha))$ the support S_w has to consist of two intervals, one containing 0 and the other containing 1. This proves (2) and we have completed the proof of the theorem. $\qquad\square$

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