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PROJECTIONS FROM $L(E,F)$ ONTO $K(E,F)$

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Abstract. Let E and F be two infinite dimensional real Banach spaces. The following question is classical and long-standing. Are the following properties equivalent?

a) There exists a projection from the space $L(E,F)$ of continuous linear operators onto the space $K(E,F)$ of compact linear operators.

b) $L(E,F) = K(E,F)$.

The answer is positive in certain cases, in particular if E or F has an unconditional basis. It seems that there are few results in the direction of a general solution. For example, suppose that E and F are reflexive and that E or F has the approximation property. Then, if $L(E,F) \neq K(E,F)$, there is no projection of norm 1, from $L(E,F)$ onto $K(E,F)$. In this paper, one obtains, in particular, the following result:

Theorem. *Let F be a real Banach space which is reflexive (resp. with a separable dual), of infinite dimension, and such that F^* has the approximation property. Let λ be a real scalar with $1 < \lambda < 2$. Then F can be equivalently renormed such that, for any projection P from $L(F)$ onto $K(F)$, one has $\|P\| \geq \lambda$. One gives also various results with two spaces E and F.*

1. Notation

All the Banach spaces in this work are real. A Banach space E is considered without special notation, as a subspace of its bidual E^{**}. For two Banach spaces E and F, we denote by $L(E,F)$ (resp. $K(E,F)$) the space of continuous (resp. compact) linear operators from E to F. If $E = F$, we simply write $L(E,F) = L(E)$, $K(E,F) = K(E)$. We denote by (AP) (resp. (MAP)) the standard notions of approximation property (resp. metric approximation property); see [7]. We denote by π (resp. ε) the projective (resp. injective) tensor norm on $E \otimes F$. For basic facts on tensor products see, for example, [9], chap. IX, 2. Let E be a Banach space and X a closed subspace of the dual E^*. We denote by $r(X)$ the characteristic of X, which is defined to be the greatest constant r such that $\sup_{x^* \in X; \|x^*\| \leq 1} |\langle x, x^* \rangle| \geq r\|x\|$, for every $x \in E$. See [2], for various properties of the characteristic.

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2. The set $G(E, M)$

Extending the notion of Godun set which is defined and studied in [4] (see also [11]), one sets the following:

Definition 2.1. Let E be a Banach space and M a closed subspace of E^{**} which contains E. We define the set $G(E, M)$ of positive scalars λ such that for any $x^{**} \in M$, there exists a net (x_b) in E which verifies the following properties:

1) $x_b \to x^{**}$ for the weak star topology $\sigma(E^{**}, E^*)$,
2) $\overline{\lim}_b \|x^{**} - \lambda x_b\| \leq \|x^{**}\|$.

By a simple exercise, one can prove that $G(E, E) = [0, 2]$. Moreover, it is clear that the function $M \to G(E, M)$ is decreasing. It follows that for any M, $G(E, M) \subset [0, 2]$.

Lemma 2.2. *Let E be a Banach space, M a subspace of E^{**} which contains E and λ a real scalar, $1 < \lambda$, such that $\lambda \in G(E, M)$. Then, for every $\varphi \in M \backslash \{0\}$, $r(\ker \varphi) \leq \frac{1}{\lambda}$. Moreover, if $M \neq E$, then there is no projection from E onto M with norm $< \lambda$.*

Proof. There exists a net (x_b) of E such that $x_b \to \varphi$ for the weak star topology $\sigma(E^{**}, E^*)$ and $\overline{\lim}_b \|\varphi - \lambda x_b\| \leq \|\varphi\|$. One has

$$r(\ker \varphi) \leq \frac{\|\frac{1}{\lambda}\varphi - x_b\|}{\|x_b\|} = \frac{1}{\lambda} \frac{\|\varphi - \lambda x_b\|}{\|x_b\|}$$

for each b (see [2]), hence

$$r(\ker \varphi) \leq \frac{1}{\lambda} \frac{\overline{\lim}_b \|\varphi - \lambda x_b\|}{\underline{\lim}_b \|x_b\|} \leq \frac{1}{\lambda}.$$

Suppose that $M \neq E$ and let P be a projection from M onto E. One has $M = E \oplus \ker P$. Let φ be an element of $\ker P \backslash \{0\}$ and $x \in E$ of norm 1. For any real α one has:

$$\begin{aligned} &x = P(x + \alpha\varphi), \\ &1 \leq \|P\| \, \|x + \alpha\varphi\|, \\ &1 \leq \|P\| r(\ker \varphi) \quad \text{(see [2])}. \end{aligned}$$

The result follows.

Proposition 2.3. *Let E be a Banach space, M a subspace of E^{**} which contains E and F a closed subspace of E. Let M_1 be a closed subspace of F^{**} which contains F. We identify F^{**} with $F^{\perp\perp}$. Then, if M_1 is a subspace of M, $G(E, M)$ is a subset of $G(F, M_1)$.*

Proof. We use a method of ([8], sublemma, p. 378). Take $\lambda \in G(E, M)$ and let φ be a non-null element of M_1. There exists a net (x_b) of E such that $x_b \to \varphi$ for the weak star topology $\sigma(E^{**}, E^*)$ and $\overline{\lim}_b \|\varphi - \lambda x_b\| \leq \|\varphi\|$. Define the following subsets:

$$\begin{aligned} U &= \{y | y \in F, \|y\| \leq \|\varphi\|\} \quad \text{and} \\ V_b &= \{t | t \in E, t \in \overline{\operatorname{conv}}_{a \geq b}(x_a)\} \quad \text{for each } b. \end{aligned}$$

It is clear that U and V_b are closed convex subsets of E. For two subsets A and B of E, $d(A, B)$ denotes the distance between A and B. Then, we will show that, for each b, $d(U, V_b) = 0$. Indeed, if $d(U, V_{b_0}) > 0$ for b_0, then there exists, by the Hahn-Banach Theorem, an $x^* \in E^*$ with $\|x^*\| = 1$ and

$$\sup_{y \in U} |\langle y, x^* \rangle| < \inf_{t \in V_{b_0}} |\langle t, x^* \rangle|.$$

By Goldstine's theorem, it follows that

$$|\langle x^*, \varphi \rangle| \leq \sup_{y \in U} |\langle y, x^* \rangle| < \inf_{t \in V_{b_0}} |\langle t, x^* \rangle| \leq |\langle x^*, \varphi \rangle|,$$

which is a contradiction. Then, there exists $y_b \in U$ and $t_b \in V_b$ for each b, so that $\|y_b - t_b\| \to 0$. One checks easily that $t_b \to \varphi$, for the weak star topology $\sigma(E^{**}, E^*)$ and it follows that $y_b \to \varphi$ for the weak star topology $\sigma(F^{**}, F^*)$. Moreover, since $t_b \in V_b$ for each b, it follows that $\overline{\lim}_b \|\varphi - \lambda t_b\| \leq \|\varphi\|$. One concludes that $\overline{\lim}_b \|\varphi - \lambda y_b\| \leq \|\varphi\|$. The result is obtained.

3. Applications to the space $K(E,F)$

Proposition 3.1. *Let F be a Banach space with a separable dual and λ a scalar with $1 < \lambda < 2$. Then F can be equivalently renormed so that*

1) *If F^* has the (AP), for any Banach space E, $\lambda \in G(K(E,F), L(E,F))$. Moreover, if $K(E,F) \neq L(E,F)$, there is no projection from $L(E,F)$ onto $K(E,F)$ with norm $< \lambda$.*

2) *For any reflexive Banach space E, $\lambda \in G(K(E,F), K(E,F)^{**})$.*

Proof. A. We suppose in this part that F has a shrinking basis, $(e_k, f_k)_{k \geq 1}$, with $e_k \in F$ and $f_k \in F^*$. Let $(S_n)_{n \geq 1}$ be the natural projections associated to the basis. By ([1], Lemma 3.4) there exists an equivalent norm on F so that $\|S_n\| = \|I - \lambda S_n\| = 1$, for each $n \geq 1$. In part A, F is equipped with this norm.

1) Let E be a Banach space. Since F^* is separable and with the (AP) one has the following equalities of Banach spaces:

$$E^* \otimes_\varepsilon F = K(E,F),$$
$$(E^* \otimes_\varepsilon F)^* = F^* \otimes_\pi E^{**}, \quad \text{and}$$
$$(E^* \otimes_\varepsilon F)^{**} = L(E^{**}, F^{**}).$$

It is clear that $E^* \otimes_\varepsilon F \subset L(E,F) \subset L(E^{**}, F^{**})$ (we consider $L(E,F)$ as a subspace of $L(E^{**}, F^{**})$ by taking the transformation $T \to T^{**}$). Let T be an element of $L(E,F)$. For any $x^{**} \in E^{**}$ and $y^* \in F^*$, one has:

$$\langle y^*, (S_n T)^{**} x^{**} \rangle = \langle (S_n T)^* y^*, x^{**} \rangle = \langle T^* S_n^* y^*, x^{**} \rangle.$$

Since $S_n^* y^*$ converges to y^* in norm, the term $\langle y^*, (S_n T)^{**} x^{**} \rangle$ converges to $\langle y^*, T^{**}, x^{**} \rangle$. But the sequence $(S_n T)$ is bounded in norm and it follows that $S_n T \to T$ for the weak star topology $\sigma(K(E,F)^{**}, K(E,F)^*)$.

Moreover, $\|T - \lambda S_n T\| \leq \|I - \lambda S_n\| \, \|T\| \leq \|T\|$, for each n.

It follows that $\lambda \in G(K(E,F), L(E,F))$. By Lemma 2.2, one deduces that if $K(E,F) \neq L(E,F)$, then there is no projection from $L(E,F)$ onto $K(E,F)$ with norm $< \lambda$.

2) Let E be a reflexive Banach space. In this case, $K(E,F)^{**} = L(E, F^{**})$. One defines, for each n, $A_n \in K(F^{**}, F)$ by $A_n(y^{**}) = \sum_{k=1}^n \langle f_k, y^{**} \rangle e_k$. Let T be an element of $L(E, F^{**})$. One checks, like in 1), that the sequence $(A_n T)$ of $K(E,F)$ converges to T for the weak star topology $\sigma(K(E,F)^{**}, K(E,F)^*)$ and also that $\|T - \lambda A_n T\| \leq \|T\|$, for each n. The result is obtained.

B. We suppose in this part that F has a separable dual. By [10], there exists a Banach space F_1 with a shrinking basis so that F can be isometrically identified with a subspace of F_1. We take the renorming of F_1 which is defined in part A. It gives a renorming of F. In all of part B we use this renorming of F.

1) Suppose that F^* has the (AP) and let E be a Banach space. One has also in this case $K(E,F)^{**} = L(E^{**}, F^{**})$. One checks easily the following inclusions of Banach spaces:

$$\begin{aligned} &K(E,F) \subset L(E,F) \subset L(E^{**}, F^{**}),\\ &K(E,F_1) \ \subset L(E,F_1) \subset L(E^{**}, F_1^{**}),\\ &K(E,F) \subset K(E,F_1), \qquad L(E,F) \subset L(E,F_1). \end{aligned}$$

Then the result follows by Lemma 2.2, Proposition 2.3 and part A of the proof.

2) Suppose that E is a reflexive Banach space. In this case by Proposition 2.3

$$G(K(E,F_1), K(E,F_1)^{**}) \subset G(K(E,F), K(E,F)^{**}).$$

Then we obtain the result by part A of the proof.

Lemma 3.2. *Let E and F be two reflexive Banach spaces such that $K(E,F)$ is nonreflexive. Then, there exists a separable subspace of F, F_1, which is one complemented in F, such that $K(E,F_1)$ is nonreflexive.*

Proof. By [3], Proposition 1.1, we can identify $K(E,F)^{**}$ with a subspace of $L(E,F)$. Let A be a noncompact element of $K(E,F)^{**}$. There exists a sequence $(x_n)_n$ of E, with $\|x_n\| = 1$, such that the sequence $(Ax_n)_n$ will be without convergent subsequence in F. Define $M = \overline{\text{span}}_n(Ax_n)$. By [6], Proposition 1, there exists a separable subspace F_1 of F which contains M and a projection R from F onto F_1 such that $\|R\| = 1$. By [3], Proposition 1.1, there exists a bounded net (T_b) of elements of $K(E,F)$ such that $T_b \to A$ for the weak star topology $\sigma(L(E,F), E\otimes_\pi F^*)$. It follows that $RT_b \to RA$ for the weak star topology $\sigma(L(E,F_1), E \otimes_\pi F_1^*)$. One deduces that RA is an element of $K(E,F_1)^{**}$. Moreover $RAx_n = Ax_n$, for all n. It follows that RA is noncompact. Then $K(E,F_1)$ is nonreflexive.

Theorem 3.3. *Let E and F be two reflexive Banach spaces such that $K(E,F)$ is nonreflexive, and let λ be a scalar, $1 < \lambda < 2$. Then F can be equivalently renormed so that, for any renorming of E, there is no projection from $L(E,F)$ onto $K(E,F)$ with norm $< \lambda$.*

Proof. Let E and F be two reflexive Banach spaces such that $K(E,F)$ is nonreflexive. By Lemma 3.2, there exists a separable subspace of F, F_1, which is one complemented in F, such that $K(E,F_1)$ is nonreflexive. Let R be the projection of F onto F_1 with $\|R\| = 1$ and i the canonical injection from F_1 to F. We denote by $\| \, \|$ the initial norm on F. Fix a real number λ with $1 < \lambda < 2$. By Proposition 3.1 there exists an equivalent norm on F_1, that we denote by $\| \, \|_1$, such that after this renorming, $\lambda \in G(K(E,F_1), K(E,F_1)^{**})$. We define an equivalent norm on F, $\| \, \|_\lambda$, by the formula $\|x\|_\lambda = \text{Max}\{\|Rx\|_1, \|(I-R)x\|\}$, $x \in F$. In the continuation of the proof F is equipped with the norm $\| \, \|_\lambda$. It is very easy to check that $\|R\|_\lambda = 1$.[1] Let P be a projection from $K(E,F)^{**}$ onto $K(E,F)$ and $U \in K(E,F_1)^{**}$. With the help of [3], Proposition 1.1, one verifies that iU is an element of $K(E,F)^{**}$. We define the transformation P_1 from $K(E,F_1)^{**}$ to $K(E,F_1)$ by $P_1(U) = R \circ P(iU)$.

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[1] $\|R\|_\lambda$ is the norm of R when F is equipped with $\| \, \|_\lambda$.

We check that P_1 is a linear projection onto $K(E, F_1)$ and also that $\lambda \leq \|P_1\|_\lambda$, by Proposition 3.1 and Lemma 2.2. One deduces that $\lambda \leq \|P\|_\lambda$. The same property follows immediately for any projection from $L(E, F)$ onto $K(E, F)$ by [3], Proposition 1.1. Moreover, the construction of the renorming of F does depend on E, but only on the isomorphic class of E and is, in fact, not affected by renorming E.

Remark 3.4. If E and F are two reflexive Banach spaces such that E or F has the (AP) and $K(E, F) \neq L(E, F)$, the conclusions of Theorem 3.3 hold since, in this case, $K(E, F)^{**} = L(E, F)$.

Theorem 3.5. *Let F be a Banach space which is reflexive (resp. with a separable dual) and of infinite dimension. Suppose that F (resp. F^*) has the (AP). Then, for any scalar λ, $1 < \lambda < 2$, F can be equivalently renormed so that there is no projection from $L(F)$ into $K(F)$ with norm $< \lambda$.*

This follows from Proposition 3.1 and Remark 3.4.

Question 3.6. Is it possible to take $\lambda = 2$ in the previous results?

Remark 3.7. The value $\lambda = 2$ seems interesting since one has the following:

Proposition 3.8. *Let E be a Banach space such that $L(E) = K(E) \oplus \operatorname{span}(I)$, and let P be the projection from $L(E)$ onto $K(E)$ with $\ker P = \operatorname{span}(I)$. Then, $\|P\| \leq 2$.*

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