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INTEGRABILITY OF SUPERHARMONIC FUNCTIONS, UNIFORM DOMAINS, AND HÖLDER DOMAINS

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Abstract. Let $S^+(D)$ denote the space of all positive superharmonic functions on a domain $D \subset \mathbf{R}^n$. Lindqvist showed that $\log S^+(D)$ is a bounded subset of $BMO(D)$. Using this, we give a characterization of finitely connected 2-dimensional uniform domains and remarks on Hölder domains.

1. Notation and main result

Let $S^+(D)$ and $H^+(D)$ denote the spaces of all positive superharmonic and positive harmonic functions on a domain $D \subset \mathbf{R}^n$, $n \geq 2$, respectively. The quasihyperbolic metric k_D on D is defined by

$$k_D(x, y) = \inf_\gamma \int_\gamma \frac{ds}{d(\cdot, \partial D)},$$

where d denotes the Euclidean distance, and the infimum is taken over all rectifiable curves $\gamma \subset D$ joining x to y (cf. [2]). We say that D is a Hölder domain if

$$k_D(x, x_0) \leq \frac{1}{\alpha} \log \left(2 + \frac{1}{d(x, \partial D)} \right) + C, \qquad x \in D,$$

for some α, $C > 0$. Note that D is a Hölder domain iff $\sup_{x \in D} \int_{B_x} e^{p k_D(\cdot, x_0)} dm < \infty$ for some $p > 0$, where B_x denotes the ball with center x and radius $d(x, \partial D)/2$, and m denotes the n-dimensional Lebesgue measure. Smith-Stegenga showed the following remarkable characterization of Hölder domains, which asserts that the local exponential integrability of the quasihyperbolic metric implies the global one:

Proposition 1.1 ([9])**.** *If D is a Hölder domain in $\mathbf{R}^n$, then $\int_D e^{p k_D(\cdot, x_0)} dm < \infty$ for some $p > 0$.*

Using this or a similar BMO argument, Smith-Stegenga, Masumoto, and Stegenga-Ullrich investigated the L^p integrability of $S^+(D)$ functions:

Proposition 1.2 ([10], [7], [11], cf. [12])**.** *If D is a Hölder domain in $\mathbf{R}^n$, then $S^+(D) \subset L^p(D)$ for some $p > 0$. Conversely, if D is a finitely connected subdomain of $\mathbf{R}^2$ and $S^+(D) \subset L^p(D)$ for some $p > 0$, then D is a Hölder domain.*

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Lindqvist clarified the argument of Stegenga-Ullrich by showing the following. Let $BMO(D)$ be the space of all locally integrable functions f on D satisfying

$$\|f\|_{*,D} = \sup |B|^{-1} \int_B |f - f_B| dm < \infty,$$

where the supremum is taken over all balls B in D, $|B| = m(B)$, and f_B denotes the integral mean of f over B.

Proposition 1.3 ([4]). *For an arbitrary subdomain D of $\mathbf{R}^n$, we have*

$$\|\log u\|_{*,D} \leq C(n), \qquad u \in S^+(D).$$

In our former paper [3], we obtained various estimations for the integrability of $BMO(D)$ functions. So, Lindqvist's theorem immediately provides the corresponding results for $S^+(D)$. Now we state one of them.

We say that a proper subdomain D of $\mathbf{R}^n$ is a uniform domain if

$$k_D(x,y) \leq C \log\left(2 + \frac{d(x,\partial D) + d(y,\partial D) + |x-y|}{\min\{d(x,\partial D), d(y,\partial D)\}}\right), \qquad x, y \in D,$$

for some $C > 0$. Each bounded uniform domain is Hölder. For $p > 0$ and a measurable subset E of $\mathbf{R}^n$, we set

$$N_p(E) = |E|^{-1} \inf_{y \in \mathbf{R}^n} \left(\int_E |x-y|^p dm(x)\right)^{\frac{n}{n+p}}.$$

$N_p(E)$ is invariant under similarities of $\mathbf{R}^n$, and a kind of distance between E and balls. Then from [3], Theorem 5.3, we have

Theorem 1.1. *If D is a uniform domain in $\mathbf{R}^n$, then there exist constants p_0, p, $C > 0$ such that for each $u \in S^+(D)$ and each measurable subset E of D, we have*

$$\left(|E|^{-1} \int_E u^p dm\right)\left(|E|^{-1} \int_E u^{-p} dm\right) \leq C N_{p_0}(E)^2.$$

Our main aim of the present paper is to show that the converse holds if D is a finitely connected subdomain of $\mathbf{R}^2$:

Theorem 1.2 (Main Theorem). *Let D be a finitely connected proper subdomain of $\mathbf{R}^2$. Assume that there exist constants p_0, p, $C > 0$ such that for each $u \in S^+(D)$ and each measurable subset E of D, we have*

$$\left(|E|^{-1} \int_E u^p dm\right)\left(|E|^{-1} \int_E u^{-p} dm\right) \leq C N_{p_0}(E)^2.$$

Then D is a uniform domain.

The proof of the Main Theorem is given in §2. In §3, we list some other immediate consequences of Lindqvist's theorem and of the author [3]. Using these results, we investigate the integrability of $S^+(D)$ functions on Hölder domains (§4) and the boundedness of domains with some integrability condition for $S^+(D)$ (§5).

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2. Proof of the Main Theorem

To prove the Main Theorem, we need a lemma, which also plays a fundamental role in §§4 and 5. From the Harnack inequality, $|\log u(x) - \log u(y)| \leq Ck_D(x,y)$, $x, y \in D$, holds for each $u \in H^+(D)$, where $C = C(n) > 0$. Conversely,

Lemma 2.1. *Let D be a finitely connected proper subdomain of $\mathbf{R}^2$. Assume that each boundary component contains more than two points. Then there exist constants C_1, $C_2 > 0$ such that for each pair of points x_1, x_2 on D, we can find a pair of an arc γ joining x_1 to x_2 and a $H^+(D)$ function u satisfying*

$$\int_{\gamma_{yz}} \frac{ds}{d(\cdot, \partial D)} \leq C_1(\log u(y) - \log u(z)) + C_2, \qquad y, z \in \gamma,$$

where γ_{yz} denotes the portion of γ joining y to z and y is between z and x_2.

Proof. Since k_D is conformally invariant modulo constant factors, we may assume that D is a bounded domain surrounded by a finite number of circles, and that x_1 and x_2 are sufficiently close to some boundary components F_1 and F_2, respectively. We may assume $F_1 \neq F_2$. A similar argument holds when $F_1 = F_2$. We may also assume that $F_1 = \{|x| = 1\}$; the outer boundary of D, and $F_2 = \{|x| = a\}$, $0 < a < 1$. Take $b > 0$ so that $\{a < |x| < a + 2b\} \subset D$, $\{1 - 2b < |x| < 1\} \subset D$. Let $x_j = r_j e^{i\theta_j}$, $j = 1, 2$. Let γ_1 and γ_2 be the segments joining x_1 to $x_1' = (1-b)e^{i\theta_1}$ and $x_2' = (a+b)e^{i\theta_2}$ to x_2, respectively. We can take an arc $\gamma' \subset D$ joining x_1' to x_2' so that $\int_{\gamma'} d(\cdot, \partial D)^{-1} ds \leq C$. Let u be the Martin kernel function for $x_2'' = ae^{i\theta_2} \in \partial D$, i.e.

$$u(x) = \lim_{D \ni y \to x_2''} (g_D(x,y)/g_D(x_0,y)),$$

where g_D is the Green function on D and x_0 is a fixed point on D. Then it is easy to check that $u(x) \approx (|x| - a)^{-1}$, $x \in \gamma_2$, $u(x) \approx 1 - |x|$, $x \in \gamma_1$, and $u(x) \approx 1$, $x \in \gamma'$. So $\gamma = \gamma_1 \cup \gamma' \cup \gamma_2$ and u satisfy the required condition. □

Proof of the Main Theorem. Assume that D satisfies the condition of the Main Theorem. In general, if D is uniform, then $D \setminus \{x\}$, $x \in D$, is also uniform. Thus we may assume that D has no punctures. Let x, $y \in D$ and set $l = |x - y|$. We may assume that $d(x, \partial D) \leq d(y, \partial D) \leq l$. Let $r = d(x, \partial D)/2$ and B_x (resp. B_y) denote the ball with center x (y) and radius r. Let $E = B_x \cup B_y$. From Lemma 2.1, there exists a $H^+(D)$ function u, $u(y) = 1$, satisfying

$$k_D(x,y) \leq C_1 \log u(x) + C.$$

Then

$$\int_E u^p dm \geq Cr^n e^{pC_1^{-1}k_D(x,y)}, \quad \int_E u^{-p} dm \geq Cr^n, \; |E| N_{p_0}(E) \leq C(l^{p_0} r^n)^{\frac{n}{n+p_0}}.$$

Hence

$$e^{pC_1^{-1}k_D(x,y)} \leq C\left(\frac{l}{r}\right)^{\frac{2np_0}{n+p_0}},$$

and so D is a uniform domain. □

3. Some other direct consequences of Lindqvist's theorem

In the present section, we list some other consequences of Lindqvist's theorem and of the author [3]. For a weight w on a ball B, we set

$$M_p(w,B)=\begin{cases}\operatorname{ess\,sup}_B w, & p=\infty,\\ \left(|B|^{-1}\displaystyle\int_B w^p dm\right)^{\frac{1}{p}}, & p\neq 0,\ p\neq\pm\infty,\\ \exp\left(|B|^{-1}\displaystyle\int_B \log w\ dm\right), & p=0,\\ \operatorname{ess\,inf}_B w, & p=-\infty.\end{cases}$$

$M_p(w,B)$ is a non-decreasing function of p. Let $\mathcal{A}(D)$ denote the space of all balls B on D satisfying $d(B,\partial D)\geq \operatorname{rad}(B)$, where $\operatorname{rad}(B)$ denotes the radius of B. We say that a weight w on a domain D satisfies the A_∞ condition locally on D ($w\in A_\infty^{\mathrm{loc}}(D)$) if $0<M_1(w,B)\leq KM_0(w,B)<\infty$, $B\in\mathcal{A}(D)$, for some $K>0$. The typical example of $A_\infty^{\mathrm{loc}}(D)$ weights are given by

$$w=d(\cdot,\partial D)^\alpha(k_D(\cdot,x_0)+1)^\beta u^\gamma,\qquad \alpha,\ \beta\in\mathbf{R},\ -\infty<\gamma<\tfrac{n}{n-2},\ u\in S^+(D).$$

Lemma 3.1. *Let $u\in S^+(D)$, $-\infty\leq p<\frac{n}{n-2}$ ($\frac{n}{n-2}=\infty$ if $n=2$), and $B\in\mathcal{A}(D)$. Then $0<M_p(u,B)\leq CM_{-\infty}(u,B)<\infty$, where $C=C(n,p)>0$.*

Proof. Let $g_D(\cdot,y)$ be the Green function on D with pole y. We may assume $1\leq p<\frac{n}{n-2}$. From the Harnack inequality, it is easy to check that $M_p(g_D(\cdot,y),B)\leq CM_{-\infty}(g_D(\cdot,y),B)$, $B\in\mathcal{A}(D)$. Since each $S^+(D)$ function can be approximated by an increasing sequence of Green potentials, we may assume that u is a Green potential of a positive measure ν on D. Then

$$M_p(u,B)\leq\int_D M_p(g_D(\cdot,y),B)d\nu(y)\leq C\int_D M_{-\infty}(g_D(\cdot,y),B)d\nu(y)\leq CM_{-\infty}(u,B).$$

□

If $w\in A_\infty^{\mathrm{loc}}(D)$, then $\log w\in BMO(D)$. So Lemma 3.1 gives another proof of Lindqvist's theorem. Let ϕ be a non-negative, non-decreasing, continuous function on $[0,\infty)$ such that $\phi(t)>0$, $t>0$. We say that ϕ is tame if $\phi(t+1)\leq C\phi(t)$, $t\geq 1$. Let $x_0\in D$, and let B_0 be the ball in D with center x_0 and radius $d(x_0,\partial D)/2$. Then from [3], Theorem 2.3, we have

Theorem 3.1. *Let ϕ be tame, D a proper subdomain of $\mathbf{R}^n$, w a $A_\infty^{\mathrm{loc}}(D)$ weight with constant factor K, and E a measurable subset of D. Assume that*

$$\int_E \phi(p_0k_D(\cdot,x_0))wdm<\infty$$

for some $p_0>0$. Then for each p, $0<p<C_1\min\{p_0,1\}$, and each $u\in S^+(D)$, we have

$$\int_E\phi(p|\log u-(\log u)_{B_0}|)wdm\leq C_2\left(\int_{B_0}wdm+\int_E\phi(p_0k_D(\cdot,x_0))wdm\right),$$

where $C_1=C_1(n,K,\phi)>0$ and $C_2=C_2(n,K,\phi,p_0)>0$.

This result has an advantage in that it gives an estimation of $S^+(D)$ functions not only from above but also from below. It is to be noted that from Lemma 3.1, we may replace the constant $(\log u)_{B_0}$ $(= \log M_0(u, B_0))$ with $\log M_p(u, B_0)$, $-\infty \le p < \frac{n}{n-2}$, in particular, with $\log(\min_{B_0} u)$ or $\log(u_{B_0})$. As to $H^+(D)$ functions, Theorem 3.1 is rather trivial, because the pointwise version $|\log u(x) - \log u(x_0)| \le Ck_D(x, x_0)$ holds by the Harnack inequality. In the case of $\phi(t) = e^t$, t^p, we have

Corollary 3.1. *Let D, w and E be as above. Assume that $\int_E e^{p_0 k_D(\cdot, x_0)} w dm < \infty$ for some $p_0 > 0$. Then for each p, $0 < p < C_1 \min\{p_0, 1\}$, we have*

$$\int_E u^{\pm p} w dm \le C_2 (u_{B_0})^{\pm p} \left(\int_{B_0} w dm + \int_E e^{p_0 k_D(\cdot, x_0)} w dm \right), \qquad u \in S^+(D),$$

where $C_1 = C_1(n, K) > 0$ and $C_2 = C_2(n, K, p_0) > 0$.

Corollary 3.2. *Let D, w and E be as above. Let $0 < p < \infty$. Assume that $\int_E k_D^p(\cdot, x_0) w dm < \infty$. Then*

$$\int_E |\log u|^p w dm \le C \left(\int_{B_0} w dm + \int_E k_D(\cdot, x_0)^p w dm \right), \qquad u \in S^+(D),\ u_{B_0} = 1,$$

where $C = C(n, K, p) > 0$.

As to the non-weighted case, from [3], Corollary 5.2, we have

Theorem 3.2. *Let D be a proper subdomain of $\mathbf{R}^n$ and E a measurable subset of D. Assume that $\int_E e^{p_0 k_D(\cdot, x_0)} dm < \infty$ for some $p_0 > 0$. Then $N_{p_0}(E) < \infty$ and for each p, $0 < p < C_1 \min\{p_0, 1\}$, and each $u \in S^+(D)$, we have*

$$\left(\int_E u^p dm \right) \left(\int_E u^{-p} dm \right) \le C_2 \left((|E| N_{p_0}(E))^2 + \left(\inf_{x_0 \in D} \int_E e^{p_0 k_D(\cdot, x_0)} dm \right)^2 \right),$$

where $C_1 = C_1(n) > 0$, and $C_2 = C_2(n, p_0) > 0$.

Note that, in general, we can omit neither the first term nor the second term on the right side of the inequality (cf. the Main Theorem).

4. Remarks on the Hölder domain

In the present section, we give some characterizations of finitely connected Hölder domains in $\mathbf{R}^2$. First, we show the following analogy of Proposition 1.2. Recall that B_0 is the disk with center x_0 and radius $d(x_0, \partial D)/2$.

Theorem 4.1. *If D is a Hölder domain in $\mathbf{R}^n$, then $\int_D u^{-p} dm \le C(u_{B_0})^{-p}$, $u \in S^+(D)$, for some p, $C > 0$. Conversely, if D is a finitely connected subdomain of $\mathbf{R}^2$ and $\int_D u^{-p} dm \le C(u_{B_0})^{-p}$, $u \in S^+(D)$, for some p, $C > 0$, then D is a Hölder domain.*

Proof. Assume that $D \subset \mathbf{R}^2$ is finitely connected and that $\int_D u^{-p} dm \le C(u_{B_0})^{-p}$, $u \in S^+(D)$, for some p, $C > 0$. Since positive constants are in $S^+(D)$, D must have finite area. In general, if D is Hölder, then $D \setminus \{x\}$, $x \in D$, is also Hölder. So we may assume that D has no punctures. Let $x \in D$. From Lemma 2.1, we can take $u \in H^+(D)$, $u(x_0) = 1$, so that $k_D(x, x_0) \le -C_1 \log u(x) + C$. Then

$$d(x, \partial D)^2 \exp(pC_1^{-1} k_D(x, x_0)) \le C \int_{B_0} u^{-p} dm \le C,$$

so D is Hölder. The remaining implication follows from Corollary 3.1 and Proposition 1.1. $\square$

Next, we consider another class of harmonic functions. Let $QLH(D)$ be the space of all harmonic, Lipschitz continuous functions h on D with respect to the quasihyperbolic metric endowed with the norm

$$\|h\|_L = \sup_{x\in D} |\nabla h(x)| d(x, \partial D).$$

Then $QLH(D)$ agrees with the space of all harmonic $BMO(D)$ functions, and the norms $\|\cdot\|_{*,D}$ and $\|\cdot\|_L$ are comparable with constant factors depending only on n. Smith-Stegenga [9] showed that a domain D is a Hölder domain iff we can take $p > 0$ so that $\int_D e^{p|f|} dm < \infty$ holds for each $f \in BMO(D)$, $\|f\|_{*,D} \le 1$. We show that we may replace $BMO(D)$ with its subspace $QLH(D)$ under some additive condition:

Theorem 4.2. *Let D be a proper subdomain of $\mathbf{R}^2$ which is conformally equivalent to some Hölder domain. Assume that there exist constants p, $C > 0$ such that*

$$\int_D e^{p|h-h(x_0)|} dm \le C, \qquad h \in QLH(D),\ \|h\|_L \le 1.$$

Then D is a Hölder domain.

Lemma 4.1. *Let D be as above. Then there exist constants C_1, $C_2 > 0$ depending only on D and x_0 such that for each $x \in D$ we can find a real $QLH(D)$ function h, $\|h\|_L \le 1$, $h(x_0) = 0$, so that $k_D(x, x_0) \le C_1 h(x) + C_2$.*

Proof. Since k_D (and so $QLH(D)$) is conformally invariant, we may assume that D is a Hölder domain from the beginning. Let $x \in D$. Take $x' \in \partial D$ so that $d(x, \partial D) = |x - x'|$. Let $h(y) = \log |y - x'|$. Then $\|h\|_L \le C$ and

$$|h(x) - h(x_0)| \ge \log \frac{1}{d(x, \partial D)} - C \ge C k_D(x, x_0) - C.$$

$\square$

Proof of Theorem 4.2. Assume that D satisfies the condition of Theorem 4.2 with $p = p_0$. Let $x \in D$. Take a real $QLH(D)$ function h satisfying the condition of the lemma above. Let $B = \{y \mid |y - x| \le d(x, \partial D)/2\}$. Then $k_D(x, x_0) \le C_1 h(y) + C$, $y \in B$, and so for $p = p_0 C_1^{-1}$, we have

$$d(x, \partial D)^2 e^{p k_D(x, x_0)} \le C \int_B e^{p_0 h} dm \le C.$$

Thus D is a Hölder domain. $\square$

Combining Proposition 1.2, and Theorems 4.1, 4.2, we have

Corollary 4.1. *For a finitely connected proper subdomain D of $\mathbf{R}^2$, the following conditions are equivalent:*
(1) D is a Hölder domain;
(2) $\int_D u^p dm \le C u(x_0)^p$, $u \in S^+(D)$, holds for some p, $C > 0$;
(3) $\int_D u^{-p} dm \le C u_{B_0}^{-p}$, $u \in S^+(D)$, holds for some p, $C > 0$;
(4) $\int_D e^{p|h-h(x_0)|/\|h\|_L} dm \le C$, $h \in QLH(D)$, holds for some p, $C > 0$.

Note that since $\log|x-x_1|$, $x_1 \in \partial D$, belongs to $QLH(D)$, if D satisfies the condition (4), then D has finite area, and so is conformally equivalent to some Hölder domain.

Finally in the present section, we give a remark on Lipschitz domains. We say that a bounded domain D in $\mathbf{R}^n$ is k-Lipschitz ($k > 0$) if D and ∂D are given locally by a Lipschitz function whose Lipschitz constant is at most k. Each Lipschitz domain is a uniform domain. For various estimations of $S^+(D)$ functions on Lipschitz domains, see Maeda-Suzuki [5], Masumoto [6], and Aikawa [1]. Let $\alpha = \alpha_n(\tan^{-1}(1/k))$, where α_n denotes the maximal order of barriers (cf. [5]). Let D be a k-Lipschitz domain. Then it is known that if

$$0 < p < \min\{n/(n+\alpha-2), 1/(\alpha-1)\},$$

then $\int_D u^p dm \le Cu(x_0)^p$, $u \in S^+(D)$, holds ([6], [1]). Moreover, from the estimation $u(x) \ge Cd(x,\partial D)^\alpha u(x_0)$, $u \in H^+(D)$, it is easy to see that if $0 < p < 1/\alpha$, then $\int_D u^{-p} dm \le Cu(x_0)^{-p}$, $u \in H^+(D)$, holds.

5. Boundedness of domains with some integrability condition

In the present section, we give another application of Lindqvist's theorem and [3]. We investigate integrability conditions for $H^+(D)$ which ensure the boundedness of D. Proposition 1.2 shows that if D is a finitely connected plane domain, then the L^p integrability of $H^+(D)$ is one such condition. Recall that B_0 is a ball with center x_0 and radius $d(x_0, \partial D)/2$.

Theorem 5.1. *Let ϕ be tame.*

(1) Let $\int_1^\infty \phi(t)^{-1} dt < \infty$. Let D be a finitely connected proper subdomain of $\mathbf{R}^2$ and at least one boundary component contain more than two points. Assume that

$$\int_D \phi(p \log^+ u)\, dm \le C_1, \qquad u \in H^+(D),\ u(x_0) = 1,$$

holds for some p, $C_1 > 0$. Then for each $x \in D$ we can take an arc γ on D joining x_0 to x so that $|\gamma| \le C_2$, where $C_2 = C_2(D, x_0, \phi, p) > 0$. In particular, D is bounded.

(1)′ In (1), we may replace $\log^+ u$ with $\log^+ \frac{1}{u}$.

(2) Conversely, let $\int_1^\infty \phi(t)^{\frac{-1}{n-1}} dt = \infty$. Then there exists an unbounded proper subdomain D of $\mathbf{R}^n$ which is homeomorphic to an open ball satisfying

$$\int_D \phi(p|\log u|)\, dm \le C, \qquad u \in S^+(D),\ u_{B_0} = 1,$$

for some p, $C > 0$.

Proof of Theorem 5.1. (2) follows from Theorem 3.1, Lindqvist's theorem, and [3], Theorem 6.1. Next, assume that ϕ and D satisfy the condition in (1) with $p = p_0$. We may assume that D has no punctures. Let $x \in D$. From Lemma 2.1, we can take a pair of an arc $\gamma : x = x(t)$, $0 \le t \le a$, joining x_0 to x and a $H^+(D)$ function u, $u(x_0) = 1$, satisfying $\int_{\gamma_y} \frac{ds}{d(\cdot, \partial D)} \le C \log u(y) + C$, $y \in \gamma$. Let $t_0 = 0$ and set $t_1 = \max\{t > t_0 \mid |x(t) - x(t_0)| \le d(x(t_0), \partial D)/2\}$. If $t_1 < a$, then set $t_2 = \max\{t > t_1 \mid |x(t) - x(t_1)| \le d(x(t_1), \partial D)/2\}$. Repeating this process, we obtain a sequence $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = a$. We may assume $k \geq 4$. Set $x_j = x(t_j)$. Since $\int_{t_j}^{t_{j+1}} d(x(t), \partial D)^{-1} |dx(t)| \geq C$, $0 \leq j \leq k-2$, we have

$$j \leq C \int_0^{t_j} \frac{|dx(t)|}{d(x(t), \partial D)} \leq C_1 \log u(x_j) + C, \qquad 0 \leq j \leq k-1.$$

Let B_j, $j \geq 2$, be the ball with center x_j and radius $r_j = d(x_j, \partial D)/10$. Then B_j, $0 \leq j \leq k-1$, are disjoint. Thus, for $p = p_0 C_1^{-1}$, we have

$$\begin{aligned}\sum_{j=2}^{k-1} r_j &\leq \left(\sum_{j=2}^{k-1} \phi(pj)^{-1} \right)^{1/2} \left(\sum_{j=2}^{k-1} \phi(pj) r_j^2 \right)^{1/2} \\ &\leq C \left(\int_p^{\infty} \phi(t)^{-1} dt \right)^{1/2} \left(\int_D \phi(p_0 \log^+ u) dm \right)^{1/2}.\end{aligned}$$

Let γ' be the associated polygon joining x_0, x_1,...,x_k. Then $|\gamma'| \leq C \sum_{j=2}^{k-1} r_j$, hence (1) follows. Finally, if we take $u \in H^+(D)$, $u(x_0) = 1$, so that $\int_{\gamma_y} \frac{ds}{d(\cdot, \partial D)} \leq -C \log u(y) + C$, $y \in \gamma$, and repeating the argument above, we get (1)$'$. $\square$

Corollary 5.1.

(1) Let $1 < p < \infty$. Let D be a finitely connected proper subdomain of $\mathbf{R}^2$ and at least one boundary component contain more than two points. Assume that

$$\int_D (\log^+ u)^p dm \leq C, \qquad u \in H^+(D),\ u_{B_0} = 1,$$

for some $C > 0$. Then D is bounded.

(1)$'$ In (1), we may replace $\log^+ u$ with $\log^+ \frac{1}{u}$.

(2) Let $0 < p \leq n-1$. Then, there exists an unbounded proper subdomain D of $\mathbf{R}^n$ which is homeomorphic to an open ball satisfying

$$\int_D |\log u|^p dm \leq C, \qquad u \in S^+(D),\ u_{B_0} = 1,$$

for some $C > 0$.

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