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# ON THE FINITE DIMENSIONAL UNITARY REPRESENTATIONS OF KAZHDAN GROUPS

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ABSTRACT. We use A. Weil's criterion to prove that all finite dimensional unitary representations of a discrete Kazhdan group are locally rigid. It follows that any such representation is unitarily equivalent to a unitary representation over some algebraic number field.

Let $\Gamma$ be a discrete group satisfying property (T) of Kazhdan, i.e. the trivial representation of $\Gamma$ is isolated in its unitary dual $\hat{\Gamma}$ (cf. [3] and [10], Ch. III); such groups will be referred to as *Kazhdan groups* in the sequel. It was shown in [15] that for a Kazhdan group $\Gamma$ every finite dimensional unitary representation is isolated in $\hat{\Gamma}$ as well, implying that in effect $\Gamma$ has only a finite number of inequivalent unitary representations in each (finite) dimension; in other words, $\Gamma$ has *finite representation type* with respect to unitary representations. This result was reproved in [16] by a different method which was subsequently used in [4] to obtain an explicit bound on the number of inequivalent $n$-dimensional unitary representations. Since most examples of Kazhdan groups are lattices in higher rank Lie groups (recently, however, examples of (apparently) different nature have been found, cf. [2], [18]) which enjoy superrigidity (cf. [10]) and hence have finite representation type with respect to all (finite dimensional) representations, it is a natural question to ask if any kind of rigidity property for finite dimensional representations can be inferred *directly* from property (T). Since an arbitrary Kazhdan group may have an infinite family of pairwise inequivalent noncompletely reducible representations (cf. Example below), one should probably ask about $SS$-rigidity, i.e. finiteness of representation type with respect to completely reducible representations. We point out that the $SS$-rigidity of Kazhdan groups would have a number of interesting applications, e.g., combined with Platonov's conjecture* on arithmeticity of $SS$-rigid linear groups (cf. [13, 7.5]) , it would imply that every linear Kazhdan group is a group of arithmetic type, i.e. it is commensurable with the direct product of a finite family of $S$-arithmetic groups (possibly, for different $S$).

Since a discrete Kazhdan group $\Gamma$ is necessarily finitely generated, one can consider the corresponding varieties $R_n(\Gamma)$ and $X_n(\Gamma)$ of $n$-dimensional (complex) representations of $\Gamma$ and of their characters (cf. [7]) ; let $\theta_\Gamma\colon R_n(\Gamma) \to X_n(\Gamma)$ be

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* *Added in proof.* Platonov's conjecture was recently disproved by Bass and Lubotzky.

the canonical morphism. Then $\Gamma$ is $SS$-rigid iff $\dim X_n(\Gamma) = 0$. Observe that the latter condition is equivalent to the fact that every point $\chi \in X_n(\Gamma)$ is isolated. In the present paper we prove the isolatedness of (classes of) unitary representations.

We recall that a representation $\rho \in R_n(\Gamma)$ is called *locally rigid* if its equivalence class $[\rho]$ is open in $R_n(\Gamma)$.

**Theorem 1.** *Let $\Gamma$ be a (discrete) Kazhdan group. Then any unitary representation $\rho\colon \Gamma \to \mathbf{U}_n$ is locally rigid. In other words, if a finite dimensional representation of a Kazhdan group is sufficiently close (on a fixed set of generators) to its unitary representation, then it is equivalent to this unitary representation.*

**Corollary 1.** *For a unitary representation $\rho\colon \Gamma \to \mathbf{U}_n$, $\theta_\Gamma(\rho)$ is an isolated point of $X_n(\Gamma)$.*

Since the complex topology on $R_n(\Gamma)$ and $X_n(\Gamma)$ is consistent with the Fell topology on $\hat{\Gamma}$, the corollary can be regarded as an extension of the isolatedness result in [15] from the space of unitary representations to the space of arbitrary representations of a fixed dimension.

*Proof of Theorem 1.* It was observed by A. Weil [17] (cf. also [7, 2.6]) that $\rho$ is locally rigid if $H^1(\Gamma, \mathfrak{g}l_n) = 0$ where $\Gamma$ acts on the Lie algebra $\mathfrak{g}l_n$ via $\mathrm{Ad} \circ \rho$, Ad being the adjoint representation of $GL_n$. To prove the vanishing of $H^1(\Gamma, \mathfrak{g}l_n)$ in our set-up, we first observe that since $\rho$ is unitary, the group $\rho(\Gamma) \subset GL_n(\mathbf{C})$ is relatively compact, and therefore $\mathfrak{g}l_n(\mathbf{C})$ admits an $\mathrm{Ad}\rho(\Gamma)$-invariant positive definite inner product. Then the required vanishing of $H^1(\Gamma, \mathfrak{g}l_n)$ is an immediate consequence of the results of Delorme and Guichardet (cf. [3], Theorem 4.7) on vanishing of $H^1(\Gamma, \mathcal{H})$ for any unitary action of a Kazhdan group $\Gamma$ on a Hilbert space $\mathcal{H}$ (for finite dimensional orthogonal representations this is proved also in [15], Theorem 5.1).

A standard consequence of Theorem 1 is the following.

**Theorem 2.** *Let $\Gamma$ be a (discrete) Kazhdan group. Any finite dimensional unitary representation $\rho\colon \Gamma \to \mathbf{U}_n$ is unitarily equivalent to a representation $\rho'\colon \Gamma \to \mathbf{U}_n$ such that*

$$\rho'(\Gamma) \subset \mathbf{U}_n \cap GL_n(K),$$

*where $K \subset \mathbf{C}$ is some algebraic number field.*

*Proof.* A standard argument shows that the isolatedness of $\theta_\Gamma(\rho)$ in $X_n(\Gamma)$ implies that in fact $\theta_\Gamma(\rho) \in X_n(\Gamma)(\overline{\mathbf{Q}})$ where $\overline{\mathbf{Q}}$ is the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$ (cf., for example, [12], 4.3). Now, we apply the functor of restriction of scalars:

$$\Theta = R_{\mathbf{C}/\mathbf{R}}(\theta_\Gamma)\colon \mathcal{R} = R_{\mathbf{C}/\mathbf{R}}(R_n(\Gamma)) \to R_{\mathbf{C}/\mathbf{R}}(X_n(\Gamma)) = \mathcal{X}.$$

There exists a $\mathbf{Q}$-defined subvariety $\mathcal{V} \subset \mathcal{R}$ such that the space of unitary representations $R(\Gamma, \mathbf{U}_n)$ is identified with the set of real points $\mathcal{V}(\mathbf{R})$. Let $\mathcal{W} = \mathcal{V} \cap \Theta^{-1}(\Theta(\rho))$. Since $\theta_\Gamma(\rho) \in X_n(\Gamma)(\overline{\mathbf{Q}})$, we have $\Theta(\rho) \in \mathcal{X}(\overline{\mathbf{Q}}_{\text{real}})$, where $\overline{\mathbf{Q}}_{\text{real}} = \overline{\mathbf{Q}} \cap \mathbf{R}$, and therefore $\mathcal{W}$ is defined over $\overline{\mathbf{Q}}_{\text{real}}$. Since the field $\overline{\mathbf{Q}}_{\text{real}}$ is real closed, well-known results for such fields (cf., for example, [5], Ch. XI) allow us to specialize $\rho \in \mathcal{W}(\mathbf{R})$ to a point $\rho' \in \mathcal{W}(\overline{\mathbf{Q}}_{\text{real}})$. Then $\rho'$ naturally corresponds to a unitary representation of $\Gamma$ such that $\theta_\Gamma(\rho') = \theta_\Gamma(\rho)$ and $\rho'(\Gamma) \subset \mathbf{U}_n \cap GL_n(\overline{\mathbf{Q}})$. Moreover, since $\Gamma$ is finitely generated, we have $\rho'(\Gamma) \subset \mathbf{U}_n \cap GL_n(K)$ for some finite field extension $K/\mathbf{Q}$.

Finally, since both representations $\rho$ and $\rho'$ are unitary, hence completely reducible, and have the same character, they are equivalent in $GL_n(\mathbf{C})$, i.e. there exists a $g \in GL_n(\mathbf{C})$ such that $\rho'(\gamma) = g^{-1}\rho(\gamma)g$ for all $\gamma \in \Gamma$. Let $g = hu$ be the polar decomposition of $g$, with $h$ hermitian and $u$ unitary (cf. [13, 3.2]). Then it follows from Lemma 3.5 in [13] that $h$ commutes with $\rho(\Gamma)$, and therefore $\rho'(\gamma) = u^{-1}\rho(\gamma)u$, as required.

*Remark.* It appears that Theorem 2 cannot be deduced from the isolatedness of $\rho$ in the space of unitary representations established earlier in [15], [16], and [4], as a unitary representation with transcendental traces may not have enough specializations among unitary representations.

Since for some common examples of Kazhdan groups (like $SL_n(\mathbf{Z})$, $n \geq 3$), any unitary representation has finite image, it may be of some interest to observe that groups constructd in Proposition 3.7 of [3] are infinite Kazhdan groups that admit a faithful unitary representation.

Speaking about possible generalizations of Theorem 1 to arbitrary representations, one should keep in mind that local rigidity may not hold for all representations of a Kazhdan group (cf. example below), so the best one can hope for is to prove that Kazhdan groups are $SS$-rigid.

**Example 1.** Let $\rho\colon G \to GL_m(\mathbf{C})$ be an irreducible rational representation of $G = SL_n(\mathbf{C})$, $n \geq 3$. Since $\rho$ is equivalent to a $\mathbf{Q}$-defined representation, there exists a lattice $\mathbf{Z}^m \subset \mathbf{C}^m$ invariant under $\rho(SL_n(\mathbf{Z}))$, so one can form the semidirect product $\Gamma(\rho) = \mathbf{Z}^m \rtimes_\rho SL_n(\mathbf{Z})$. It follows from Theorem 4.2 in [15] that for any nontrivial irreducible $\rho$, the group $\mathcal{G}(\rho) = \mathbf{R}^m \rtimes_\rho SL_n(\mathbf{R})$ is Kazhdan. So, being a lattice in $\mathcal{G}(\rho)$, $\Gamma(\rho)$ is also Kazhdan. Now we are going to show that it is possible to specialize $\rho$ in such a way that $\Gamma(\rho)$ would have infinite representation type. Since $\Gamma(\rho)$ is Zariski dense in $G(\rho) = \mathbf{C}^m \rtimes_\rho SL_n(\mathbf{C})$, it suffices to show that $G(\rho)$ can have an infinite family of pairwise inequivalent rational representations in some dimension, for an appropriate $\rho$. Our choice of $\rho$ will be subject to one condition: there should exist two inequivalent irreducible representations $\rho_1$ and $\rho_2$ of $G$ such that $\rho_1 \otimes \rho_2^*$ contains $\rho$ with multiplicity $> 1$, where $^*$ denotes the contragredient representation (we thank E.B. Vinberg for explaining to us how to construct such $\rho$ using the formula for multiplicities, cf. [11], p. 290, and for pointing out that the standard $n$-dimensional representation of $G$ doesn't have this property). Let $n_i = \dim \rho_i$, and $M_{n_1 n_2}(\mathbf{C})$ be the space of $(n_1 \times n_2)$-matrices with the following action of $SL_n(\mathbf{C})$ :

$$g \cdot A = \rho_1(g) A \rho_2(g)^{-1}.$$

Obviously, this representation of $SL_n(\mathbf{C})$ is equivalent to $\rho_1 \otimes \rho_2^*$, so by our construction one can find two embeddings $f, g\colon (\mathbf{C}^m, \rho) \to M_{n_1 n_2}(\mathbf{C})$ of $SL_n(\mathbf{C})$-modules with images having zero intersection. We let $h_a = af + g$ $(a \in \mathbf{C})$ and define a representation $\pi_a\colon \mathbf{C}^m \rtimes SL_n(\mathbf{C}) \to GL_{n_1+n_2}(\mathbf{C})$ by:

$$\pi_a((v,x)) = \begin{pmatrix} \rho_1(x) & h_a(v)\rho_2(x) \\ 0 & \rho_2(x) \end{pmatrix}.$$

If $\pi_b = T\pi_a T^{-1}$, then $T$ commutes with $\rho_1 \oplus \rho_2$, implying that

$$T = \operatorname{diag}(\alpha E_{n_1},\ \beta E_{n_2})$$

for some $\alpha, \beta \in \mathbf{C}^*$. In this case, from

$$T\pi_a(v)T^{-1} = \begin{pmatrix} E_{n_1} & \alpha\beta^{-1}h_a(v) \\ 0 & E_{n_2} \end{pmatrix} = \pi_b(v)$$

one obtains $a = b$, so representations $\{\pi_a \mid a \in \mathbf{C}\}$ are pairwise inequivalent.

We observe that this example provides also a group $\Gamma$ with bounded generation which satisfies the condition:

$(*)$ for any subgroup $\Gamma_1 \subset \Gamma$ of finite index, the abelianization ${\Gamma_1}^{\mathrm{ab}} = \Gamma_1/[\Gamma_1, \Gamma_1]$ is finite,

but has infinite representation type ($SS$-rigidity of such groups was established in [14]).

In fact, condition (*) is *necessary* for $\Gamma$ to have finite representation type, or even to be $SS$-rigid. It is automatically satisfied by all Kazhdan groups, and more generally, by groups satisfying property $(\tau)$ introduced by Lubotzky and Zimmer (for convenience of the reader we recall that $(\tau)$ means that the trivial representation of $\Gamma$ is isolated in the unitary dual from all representations with finite image, cf. [6], [9]). We have been able to obtain only partial generalization of Theorem 1 for groups satisfying (*).

**Proposition 1.** *Let $\Gamma$ be a finitely generated group satisfying (*). Then any representation $\rho \in R_n(\Gamma)$ with finite image is locally rigid.*

*Proof.* Let $\Delta = \operatorname{Ker}\colon \rho$. We have the following exact sequence:

$$0 \to H^1(\Gamma/\Delta, \mathfrak{g}l_n^{\Delta}) \to H^1(\Gamma, \mathfrak{g}l_n) \to H^1(\Delta, \mathfrak{g}l_n)$$

where $\Gamma$ acts on $\mathfrak{g}l_n$ via $\operatorname{Ad} \circ \rho$. By our construction $\Delta$ acts trivially, so

$$H^1(\Delta, \mathfrak{g}l_n) = \operatorname{Hom}(\Delta, \mathfrak{g}l_n) = 0,$$

as ${\Delta}^{\mathrm{ab}}$ is finite. On the other hand, since all representations of $\Gamma/\Delta$ are completely reducible, $H^1(\Gamma/\Delta, \mathfrak{g}l_n^{\Delta}) = H^1(\Gamma/\Delta, \mathfrak{g}l_n) = 0$ (cf. [7, p. 37], it immediately follows from the fact that $H^1(\Gamma/\Delta, \mathfrak{g}l_n) = \operatorname{Ext}_{\Gamma/\Delta}(\mathbf{C}^n, \mathbf{C}^n)$). Thus, $H^1(\Gamma, \mathfrak{g}l_n) = 0$ and $\rho$ is locally rigid by Weil's criterion.

This proposition, in particular, implies that there are only finitely many equivalence classes of representations of $\Gamma$ with finite image in each dimension. It turns out, however, that finiteness (up to conjugacy) of Zariski closures can be proved for arbitrary completely reducible representations.

**Proposition 2.** *Let $\Gamma$ be a group satisfying (*). For every $n \geq 1$, there exists a finite collection $G_1, \dots, G_d$ of algebraic subgroups of $GL_n(\mathbf{C})$ such that for any completely reducible representation $\rho\colon \Gamma \to GL_n(\mathbf{C})$, the Zariski closure $G = \overline{\rho(\Gamma)}$ is conjugate to one of $G_i$'s.*

*Proof.* Since $\rho$ is completely reducible, $G$ is reductive. Moreover, the fact that for $\Delta = \rho^{-1}(\rho(\Gamma) \cap G^\circ)$ the abelianization ${\Delta}^{\mathrm{ab}}$ is finite, implies that $G^\circ$ is actually semi-simple. Since there are finitely many conjugacy classes of connected semi-simple subgroups $S \subset GL_n$, it suffices to prove that, up to conjugacy, there are finitely many possibilities for $G$ with fixed connected component $G^\circ$. However, it was shown by Bass [1] that the index $[G : G^\circ]$ of the connected component is bounded by a constant $\delta_\Gamma(n)$ depending only on $n$ (but not on $\rho$). So, our claim readily follows from the fact that the finite subgroups of a given order in any algebraic group in characteristic zero fall into a finite number of conjugacy

classes (cf. [13, 2.17]), applied to the quotient $N/G^\circ$ where $N = N_{GL_n}(G^\circ)$ is the normalizer of $G^\circ$.

*Remark.* A. Lubotzky suggested that for groups satisfying $(\tau)$ it should be possible to obtain an explicit expression for the function $\delta_\Gamma(n)$ involved in the proof of Proposition 2 using the following result in [8], Theorem 3.6: if $\Gamma$ satisfies property $(\tau)$, then there exists a constant $C$ such that $\mid \Delta^{\text{ab}} \mid \leq C^{[\Gamma:\Delta]}$ for any subgroup $\Delta \subset \Gamma$ of finite index. This expression (which plays a role similar to that of the explicit estimation in [4]) comes out to be of the form:

$$\delta_\Gamma(n) = C_1^{(49n)^{n^2}\cdot n!} \quad \text{where} \quad C_1 = 2C.$$

To show this, we first observe that the image $R$ of the normalizer $N = N_{GL_n}(G^\circ)$ in $\text{Out}(G^\circ)$ has order $\leq n!$ (indeed, if $T \subset G^\circ$ is a maximal torus then every element of $R$ has a representative in $N \cap N_{GL_n}(T)$, and since any automorphism of $G^\circ$ acting trivially on $T$ is inner, we obtain that $\mid R \mid \leq \mid N_{GL_n}(T)/C_{GL_n}(T) \mid \leq n!$ where $C_{GL_n}(T)$ is the centralizer of $T$). It follows that for the centralizer $Z = C_G(G^\circ)$ we have $[G : ZG^\circ] \leq n!$. Being finite, $Z$ by Jordan's theorem contains an abelian normal subgroup $A \subset Z$ of index $\leq (49n)^{n^2}$. Then $[G : AG^\circ] \leq f(n) = (49n)^{n^2} \cdot n!$, and

$$[G : G^\circ] \leq f(n) \cdot (\rho^{-1}(\rho(\Gamma) \cap AG^\circ))^{\text{ab}} \leq f(n) \cdot C^{f(n)} \leq (2C)^{f(n)}.$$

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