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# HAMILTONIAN STATIONARY NORMAL BUNDLES OF SURFACES IN $\mathbf{R}^3$

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*Dedicated to Professor Shukichi Tanno on his* 60*th birthday*

Abstract. A surface in $\mathbf{R}^3$ has Hamiltonian stationary normal bundle if and only if it is either minimal, a part of a round sphere, or a part of a cone with vertex angle $\pi/2$.

## 0. Introduction

Let $N$ be an $n$-dimensional Kähler manifold with the Kähler form $\Omega$. A Lagrangian submanifold $M$ in $N$ is an $n$-dimensional submanifold in $N$ such that $\Omega|_M = 0$. A Lagrangian submanifold in a Kähler manifold is called Hamiltonian stationary if it is a critical point of the volume functional for all Hamiltonian deformations (cf. [2]). Of course, any minimal Lagrangian submanifold is Hamiltonian stationary. In fact, a Lagrangian submanifold with parallel mean curvature is Hamiltonian stationary (cf. Section 1).

On the other hand, let $M$ be a submanifold in $\mathbf{R}^n$. The normal bundle $T^\perp M$ of $M$ may be naturally immersed in $\mathbf{R}^n \times \mathbf{R}^n$ by the immersion $\psi\colon T^\perp M \to \mathbf{R}^n \times \mathbf{R}^n$ defined by $\psi(\nu_x) = (x, \nu_x)$. We consider the complex structure $J$ on $C^n = \mathbf{R}^n \times \mathbf{R}^n$ defined by $J(X,Y) = (-Y, X)$. Then it is known that $\psi(T^\perp M)$ is a Lagrangian submanifold in $C^n = \mathbf{R}^n \times \mathbf{R}^n$ (cf. [1, III.3.C]).

So it is natural to ask which submanifolds in $\mathbf{R}^n$ have Hamiltonian stationary normal bundles. In this paper we discuss the case of surfaces in $\mathbf{R}^3$.

**Theorem.** *Let $S$ be a surface in $\mathbf{R}^3$. Then $\psi(T^\perp S)$ is Hamiltonian stationary if and only if $S$ is either minimal, a part of a round sphere, or a part of a cone with vertex angle $\pi/2$.*

In the proof of the theorem, we can find that the normal bundles of a round sphere and a cone with vertex angle $\pi/2$ in $\mathbf{R}^3$ have non-parallel mean curvature.

*Remark.* Harvey and Lawson determined submanifolds in $\mathbf{R}^n$ with minimal normal bundles (see [1, III. Th. 3.11, Prop. 2.17]). In particular, they showed that a surface $S$ in $\mathbf{R}^n$ has minimal normal bundle if and only if $S$ is minimal.

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## 1. Preliminaries

Let $N$ be an $n$-dimensional Kähler manifold with the complex structure $J$ and the Kähler metric $g$. The Kähler form $\Omega$ of $N$ is defined by $\Omega(X,Y) = g(JX,Y)$ for $X$, $Y \in T_xN$. Let $M$ be a Lagrangian submanifold in $N$, that is, an $n$-dimensional submanifold in $N$ such that $\Omega|_M = 0$. Then $J$ becomes a bijection between $T_xM$ and $T_x^\perp M$ for $x \in M$.

A normal vector field $V$ along $M$ is called a Hamiltonian variation if $V = J(\mathrm{grad}(f))$ for some compactly supported function $f$ on $M$. Let $i\colon M \to N$ be the inclusion map. A compactly supported deformation $\phi_t\colon M \to N$ $(-\varepsilon < t < \varepsilon,\ \phi_0 = i)$ of $M$ is called a Hamiltonian deformation if its variation vector field is a Hamiltonian variation. We say that $M$ is Hamiltonian stationary if it satisfies

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(\phi_t(M)) = 0$$

for all Hamiltonian deformations $\phi_t$. The Euler-Lagrange equation is given as follows:

**Proposition** (cf. [2])**.** *Let $N$ be a Kähler manifold with the complex structure $J$. A Lagrangian submanifold $M$ in $N$ is Hamiltonian stationary if and only if its mean curvature vector $H$ satisfies* $\mathrm{div}(JH) = 0$ *on $M$.*

From this proposition, we can see that a Lagrangian submanifold with parallel mean curvature in a Kähler manifold is Hamiltonian stationary.

## 2. Proof of the Theorem

Let $x = x(t_1,t_2)$, $(t_1,t_2) \in D$, be a local parametrization of $S$, where $D$ is an open domain on $\mathbf{R}^2$. Let

$$\mathrm{I} = E(dt_1)^2 + 2F dt_1\, dt_2 + G(dt_2)^2$$

and

$$\mathrm{II} = L(dt_1)^2 + 2M dt_1\, dt_2 + N(dt_2)^2$$

be the first and second fundamental forms of $S$, respectively. As our argument is local in nature, we may assume that either $S$ has no umbilic points or $S$ is totally umbilic. In the case where $S$ has no umbilic points, we may choose the local parametrization so that the parameter curves are lines of curvature, and $F = M = 0$. It is possible to choose the local parametrization such that $F = M = 0$ also in the case where $S$ is totally umbilic. So we assume that $F = M = 0$ in the following.

The principal curvatures $a$ and $b$ of $S$ are given by $a = L/E$ and $b = N/G$. By the Codazzi equation, we have

$$(1) \qquad \frac{\partial a}{\partial t_2} = \frac{b-a}{2E}\frac{\partial E}{\partial t_2}, \qquad \frac{\partial b}{\partial t_1} = \frac{a-b}{2G}\frac{\partial G}{\partial t_1}.$$

Let $\nu$ denote the unit normal vector along $S$. Let $f\colon D \times R \to \mathbf{R}^3 \times \mathbf{R}^3$ be defined by

$$f(t_1,t_2,t_3) = (x(t_1,t_2), t_3\nu(t_1,t_2)),$$

which is the parametrization of the immersion $\psi\colon T^\perp S \to \mathbf{R}^3 \times \mathbf{R}^3$ in the introduction.

Set

$$\begin{aligned} e_1 &= (E(1+t_3^2a^2))^{-1/2}\frac{\partial}{\partial t_1}, \\ e_2 &= (G(1+t_3^2b^2))^{-1/2}\frac{\partial}{\partial t_2}, \qquad e_3 = \frac{\partial}{\partial t_3}. \end{aligned} \tag{2}$$

Then we have

$$\begin{aligned} f_*e_1 &= (E(1+t_3^2a^2))^{-1/2}\left(\frac{\partial x}{\partial t_1}, -t_3a\frac{\partial x}{\partial t_1}\right), \\ f_*e_2 &= (G(1+t_3^2b^2))^{-1/2}\left(\frac{\partial x}{\partial t_2}, -t_3b\frac{\partial x}{\partial t_2}\right), \qquad f_*e_3 = (0,\nu), \end{aligned} \tag{3}$$

which are orthonormal. So $\{e_1, e_2, e_3\}$ is an orthonormal frame on $D \times R$ with respect to the metric induced by $f$.

Let $J$ be the complex structure on $C^3 = \mathbf{R}^3 \times \mathbf{R}^3$ defined by $J(X,Y) = (-Y,X)$. Set

$$\begin{aligned} e_4 &= J(f_*e_1) = (E(1+t_3^2a^2))^{-1/2}\left(t_3a\frac{\partial x}{\partial t_1}, \frac{\partial x}{\partial t_1}\right), \\ e_5 &= J(f_*e_2) = (G(1+t_3^2b^2))^{-1/2}\left(t_3b\frac{\partial x}{\partial t_2}, \frac{\partial x}{\partial t_2}\right), \\ e_6 &= J(f_*e_3) = (-\nu, 0). \end{aligned} \tag{4}$$

Then $\{e_4, e_5, e_6\}$ is a normal orthonormal frame.

The second fundamental form $h_{ij}^\alpha$ of $f$ is defined by $h_{ij}^\alpha = \langle e_i(f_*e_j), e_\alpha\rangle$ for $1 \le i$, $j \le 3$, $4 \le \alpha \le 6$. Using (2), (3), (4) and (1), we get

$$\begin{aligned} h_{11}^4 &= -t_3E^{-1/2}(1+t_3^2a^2)^{-3/2}\frac{\partial a}{\partial t_1}, \\ h_{22}^4 &= -t_3E^{-1/2}(1+t_3^2a^2)^{-1/2}(1+t_3^2b^2)^{-1}\left(\frac{a-b}{2G}\frac{\partial G}{\partial t_1}\right) \\ &= -t_3E^{-1/2}(1+t_3^2a^2)^{-1/2}(1+t_3^2b^2)^{-1}\frac{\partial b}{\partial t_1}, \\ h_{11}^5 &= -t_3G^{-1/2}(1+t_3^2a^2)^{-1}(1+t_3^2b^2)^{-1/2}\left(\frac{b-a}{2E}\frac{\partial E}{\partial t_2}\right) \\ &= -t_3G^{-1/2}(1+t_3^2a^2)^{-1}(1+t_3^2b^2)^{-1/2}\frac{\partial a}{\partial t_2}, \\ h_{22}^5 &= -t_3G^{-1/2}(1+t_3^2b^2)^{-3/2}\frac{\partial b}{\partial t_2}, \\ h_{11}^6 &= -a(1+t_3^2a^2)^{-1}, \qquad h_{22}^6 = -b(1+t_3^2b^2)^{-1}, \\ h_{33}^4 &= h_{33}^5 = h_{33}^6 = 0. \end{aligned}$$

Set

$$\begin{aligned} P &= E^{-1/2}(1+t_3^2a^2)^{-3/2}\frac{\partial a}{\partial t_1} \\ &\quad + E^{-1/2}(1+t_3^2a^2)^{-1/2}(1+t_3^2b^2)^{-1}\frac{\partial b}{\partial t_1}, \\ Q &= G^{-1/2}(1+t_3^2a^2)^{-1}(1+t_3^2b^2)^{-1/2}\frac{\partial a}{\partial t_2} \\ &\quad + G^{-1/2}(1+t_3^2b^2)^{-3/2}\frac{\partial b}{\partial t_2}, \\ R &= a(1+t_3^2a^2)^{-1} + b(1+t_3^2b^2)^{-1}. \end{aligned} \tag{5}$$

Then the mean curvature vector $H$ of $f$ is given by

$$H = -(t_3Pe_4 + t_3Qe_5 + Re_6). \tag{6}$$

By (6) and (4),

$$JH = t_3P(f_*e_1) + t_3Q(f_*e_2) + R(f_*e_3),$$

and we have

$$\begin{aligned} \operatorname{div}(JH) &= \sum_{i=1}^{3}\langle (e_i(JH))^T,\ f_*e_i\rangle = \sum_{i=1}^{3}\langle e_i(JH),\ f_*e_i\rangle \\ &= t_3(e_1P) + t_3Q\langle e_1(f_*e_2),\ f_*e_1\rangle + R\langle e_1(f_*e_3),\ f_*e_1\rangle \\ &\quad + t_3P\langle e_2(f_*e_1),\ f_*e_2\rangle + t_3(e_2Q) + R\langle e_2(f_*e_3),\ f_*e_2\rangle \\ &\quad + t_3P\langle e_3(f_*e_1),\ f_*e_3\rangle + t_3Q\langle e_3(f_*e_2),\ f_*e_3\rangle + e_3R. \end{aligned} \tag{7}$$

Here $(e_i(JH))^T$ denotes the tangential part of $e_i(JH)$, and the seventh and eighth terms of the right hand side are zero.

(i) If $S$ is minimal, then $a + b = 0$. By (5) and (6) we have $H = 0$, and $\operatorname{div}(JH) = 0$.

(ii) If $S$ is a part of a round sphere, then $a = b$ which is constant. By (5), $P = Q = 0$ and $R = 2a(1+t_3^2a^2)^{-1}$. Using (7), (2) and (3), we can see that $\operatorname{div}(JH) = 0$.

(iii) In what follows, we assume that $S$ is neither minimal nor a part of a round sphere. Then we may assume that $a^2 \neq b^2$.

(iii)$_1$ We assume further that $S$ is non-flat. Then we may assume that $a \neq 0$ and $b \neq 0$. Using (7), (5), (2) and (3), we can find that

$$\begin{aligned} &(1+t_3^2a^2)^3(1+t_3^2b^2)^3\operatorname{div}(JH) \\ &= t_3\left\{(1+t_3^2a^2)A_1 - \frac{3a}{E}\left(\frac{\partial a}{\partial t_1}\right)^2 t_3^2(1+t_3^2b^2)^3\right\} \qquad (8) \\ &= t_3\left\{(1+t_3^2b^2)B_1 - \frac{3b}{G}\left(\frac{\partial b}{\partial t_2}\right)^2 t_3^2(1+t_3^2a^2)^3\right\} \qquad (9) \end{aligned}$$

for some functions $A_1$, $B_1$ on $D \times R$, which are polynomial with respect to $t_3$.

If $\operatorname{div}(JH) = 0$, then (8) and (9) are identically zero, which are true also as polynomials for $t_3 \in C$. So by letting $t_3 = \sqrt{-1}/a$ and $\sqrt{-1}/b$ in (8) and (9) respectively, we get

$$\frac{\partial a}{\partial t_1} = \frac{\partial b}{\partial t_2} = 0. \tag{10}$$

By (5) and (10),

$$\begin{aligned} P &= E^{-1/2}(1+t_3^2a^2)^{-1/2}(1+t_3^2b^2)^{-1}\frac{\partial b}{\partial t_1}, \\ Q &= G^{-1/2}(1+t_3^2a^2)^{-1}(1+t_3^2b^2)^{-1/2}\frac{\partial a}{\partial t_2}. \end{aligned} \tag{11}$$

Again by using (7), (11), (5), (2), (3) and (1), we have

$$\begin{aligned} &(1+t_3^2a^2)^2(1+t_3^2b^2)^2 \operatorname{div}(JH) \\ &= t_3(1+t_3^2a^2)A_2 + t_3(1+t_3^2b^2)\left\{\frac{1+a(2a-b)t_3^2}{(b-a)G}\left(\frac{\partial a}{\partial t_2}\right)^2 - a^3(1+t_3^2b^2)\right\} \quad (12) \\ &= t_3(1+t_3^2b^2)B_2 + t_3(1+t_3^2a^2)\left\{\frac{1+b(2b-a)t_3^2}{(a-b)E}\left(\frac{\partial b}{\partial t_1}\right)^2 - b^3(1+t_3^2a^2)\right\} \quad (13) \end{aligned}$$

for some functions $A_2$ and $B_2$ on $D \times R$, which are polynomials with respect to $t_3$. As we assume that $\operatorname{div}(JH) = 0$, (12) and (13) are identically zero, also as polynomials for $t_3 \in C$. So by letting $t_3 = \sqrt{-1}/a$ and $\sqrt{-1}/b$ in (12) and (13) respectively, we get

$$G = \frac{1}{a^2(a^2-b^2)}\left(\frac{\partial a}{\partial t_2}\right)^2 \neq 0, \tag{14}$$

$$E = \frac{1}{b^2(b^2-a^2)}\left(\frac{\partial b}{\partial t_1}\right)^2 \neq 0. \tag{15}$$

Inserting (14) into (1), and noting (10), (15), we have a contradiction.

Thus in this case (iii)$_1$, $\operatorname{div}(JH)$ cannot be identically zero.

(iii)$_2$ We assume that $S$ is flat. Then we may assume that $a = 0$ and $b \neq 0$ on $S$. By (5),

$$\begin{aligned} P &= E^{-1/2}(1+t_3^2b^2)^{-1}\frac{\partial b}{\partial t_1}, \\ Q &= G^{-1/2}(1+t_3^2b^2)^{-3/2}\frac{\partial b}{\partial t_2}, \qquad R = b(1+t_3^2b^2)^{-1}. \end{aligned} \tag{16}$$

Using (7), (16), (2) and (3), we have

$$(1+t_3^2b^2)^3 \operatorname{div}(JH) = t_3\left\{(1+t_3^2b^2)B_3 - \frac{3b}{G}\left(\frac{\partial b}{\partial t_2}\right)^2 t_3^2\right\} \tag{17}$$

for some function $B_3$ on $D \times R$, which is a polynomial with respect to $t_3$.

If $\operatorname{div}(JH) = 0$, then the right side of (17) is identically zero, also as a polynomial for $t_3 \in C$. Thus we have $\partial b/\partial t_2 = 0$. So $Q = 0$ by (16). Again by using (7), (16), (2), (3) and (1), we get

$$\begin{aligned}\operatorname{div}(JH) = t_3 E^{-1} \Bigg[ & b^{-1}(1+t_3^2 b^2)^{-2} \left\{ \left(\frac{\partial b}{\partial t_1}\right)^2 - E b^4 \right\} \\ & + (1+t_3^2 b^2)^{-1} \left\{ \frac{\partial^2 b}{\partial t_1^2} - \frac{2}{b}\left(\frac{\partial b}{\partial t_1}\right)^2 - \frac{1}{2E}\frac{\partial E}{\partial t_1}\frac{\partial b}{\partial t_1} \right\} \Bigg]. \end{aligned} \tag{18}$$

As we assume that $\operatorname{div}(JH) = 0$, by (18), we have

$$E = \frac{1}{b^4}\left(\frac{\partial b}{\partial t_1}\right)^2 \neq 0, \tag{19}$$

$$\frac{\partial^2 b}{\partial t_1^2} - \frac{2}{b}\left(\frac{\partial b}{\partial t_1}\right)^2 - \frac{1}{2E}\frac{\partial E}{\partial t_1}\frac{\partial b}{\partial t_1} = 0, \tag{20}$$

where we note that (19) implies (20) automatically. By (1) and that $b$ depends only on $t_1$,

$$G = \frac{c}{b^2}, \qquad N = \frac{c}{b}$$

for some positive function $c$ depending only on $t_2$. Thus we have

$$\mathrm{I} = \frac{1}{b^4}\left(\frac{\partial b}{\partial t_1}\right)^2 (dt_1)^2 + \frac{c}{b^2}(dt_2)^2, \qquad \mathrm{II} = \frac{c}{b}(dt_2)^2.$$

Noting that $c > 0$ and $\partial b/\partial t_1 \neq 0$, we change the parameters as follows:

$$\tilde{t}_1 = \frac{1}{b}, \qquad \tilde{t}_2 = \int \sqrt{c}\, dt_2.$$

Then we have

$$\mathrm{I} = (d\tilde{t}_1)^2 + \tilde{t}_1^2 (d\tilde{t}_2)^2, \qquad \mathrm{II} = \tilde{t}_1 (d\tilde{t}_2)^2. \tag{21}$$

The cone parametrized by

$$x(t_1, t_2) = \frac{1}{\sqrt{2}}(t_1 \cos(\sqrt{2} t_2), t_1 \sin(\sqrt{2} t_2), t_1) \tag{22}$$

has the first and second fundamental forms (21) without tilde, with respect to the suitable unit normal vector field. So by the fundamental theorem, up to congruence, $S$ must be a part of the cone with vertex angle $\pi/2$.

Conversely, let $S$ be a part of a cone with vertex angle $\pi/2$. Then we may assume that $S$ is parametrized by (22), and (21) without tilde is valid. With respect to this parametrization, (18) may be applied, where $E = 1$ and $b = 1/t_1$. By (18) we can see that $S$ has Hamiltonian stationary normal bundle.

Thus the proof is complete.

## References

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