

ON COVERING MULTIPLICITY

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ABSTRACT. Let $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ be a system of arithmetic sequences which forms an m -cover of \mathbb{Z} (i.e. every integer belongs at least to m members of A). In this paper we show the following surprising properties of A : (a) For each $J \subseteq \{1, \dots, k\}$ there exist at least m subsets I of $\{1, \dots, k\}$ with $I \neq J$ such that $\sum_{s \in I} 1/n_s - \sum_{s \in J} 1/n_s \in \mathbb{Z}$. (b) If A forms a minimal m -cover of \mathbb{Z} , then for any $t = 1, \dots, k$ there is an $\alpha_t \in [0, 1)$ such that for every $r = 0, 1, \dots, n_t - 1$ there exists an $I \subseteq \{1, \dots, k\} \setminus \{t\}$ for which $|\sum_{s \in I} 1/n_s| \geq m - 1$ and $\{\sum_{s \in I} 1/n_s\} = (\alpha_t + r)/n_t$.

1. INTRODUCTION

For integer a and positive integer n we call

$$a(n) = \{x \in \mathbb{Z} : x \equiv a \pmod{n}\} = a + n\mathbb{Z}$$

an *arithmetic sequence* with *common difference* n or a *residue class* with *modulus* n . For a finite system

$$(1) \quad A = \{a_s(n_s)\}_{s=1}^k$$

of such sets, we define its *covering multiplicity* by

$$(2) \quad m(A) = \inf_{x \in \mathbb{Z}} |S(x)|$$

where $S(x) = \{1 \leq s \leq k : x \equiv a_s \pmod{n_s}\}$. It is easy to show that

$$(3) \quad \sum_{s=1}^k \frac{1}{n_s} \geq m(A),$$

and the equality holds if and only if (1) covers each integer exactly m times for some $m = 1, 2, 3, \dots$. (Cf. [S2], [S4].)

Let m be a nonnegative integer. If system (1) has covering multiplicity at least m , then we call (1) an m -cover (of \mathbb{Z}). A *minimal m -cover* (of \mathbb{Z}) is an m -cover whose proper subsystems are not. If $|S(x)| = m$ for all $x \in \mathbb{Z}$, then we say that A forms an *exact m -cover* (of \mathbb{Z}). Notice that an exact 1-cover is a partition of \mathbb{Z} into (finitely many) periodic sets. The Chinese Remainder Theorem tells that the intersection of residue classes $a_1(n_1), \dots, a_k(n_k)$ is empty if and only if two of them are disjoint. So, as a dual question, when (1) forms a 1-cover is fundamental and

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important. In fact, 1-covers and exact m -covers (especially exact 1-covers) have been investigated for many years; also some famous conjectures remain open. (See R. K. Guy [G].)

Now we introduce some notation. As usual, if m and n are integers, then (m, n) represents the greatest common divisor of m and n . For a real number x , we set $\binom{x}{0} = 1$ and let $\binom{x}{n} = \prod_{j=0}^{n-1} \frac{x-j}{n-j}$ for $n = 1, 2, 3, \dots$; also $[x]$ and $\{x\}$ denote the integral and the fractional parts of x respectively.

In this paper we study the covering multiplicity of a general system of residue classes. Our main result is as follows.

Theorem 1. *Let (1) be a system of arithmetic sequences, and let J be a subset of $\{1, \dots, k\}$. Put $J^- = \{1, \dots, k\} \setminus J$.*

(i) *For any $m_1, \dots, m_k \in \mathbb{Z}$ we have*

$$(4) \quad \left| \left\{ I \subseteq \{1, \dots, k\} : I \neq J \text{ \& } \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} = \left\{ \sum_{s \in J} \frac{m_s}{n_s} \right\} \right\} \right| \geq m(A).$$

(ii) *Suppose $\emptyset \neq J \subseteq S(x)$ for some $x \in \mathbb{Z}$ with $|S(x)| = m(A)$. For each $s \in J^-$ let m_s be a positive integer prime to n_s . Then there exists an $\alpha \in [0, 1)$ such that*

$$(5) \quad \begin{aligned} & \left| \left\{ \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} : I \subseteq J^-, \left[\sum_{s \in I} \frac{m_s}{n_s} \right] \geq m(A) - |J| \right\} \right| \\ & \supseteq \left\{ \frac{a}{N(J)} : 0 \leq a < N(J), \{a\} = \alpha \right\}, \end{aligned}$$

where $N(J)$ denotes the least common multiple of those n_s with $s \in J$.

In view of Theorem 1, an m -cover $A = \{a_s(n_s)\}_{s=1}^k$ possesses the following properties:

(a) For each $J \subseteq \{1, \dots, k\}$, there exist at least m subsets I of $\{1, \dots, k\}$ with $I \neq J$ such that $\sum_{s \in I} 1/n_s - \sum_{s \in J} 1/n_s \in \mathbb{Z}$.

(b) If A forms a minimal m -cover of \mathbb{Z} , then for any $t = 1, \dots, k$ there is an $\alpha_t \in [0, 1)$ such that, for every $r = 0, 1, \dots, n_t - 1$, there exists an $I \subseteq \{1, \dots, k\} \setminus \{t\}$ for which $[\sum_{s \in I} 1/n_s] \geq m - 1$ and $\{\sum_{s \in I} 1/n_s\} = (\alpha_t + r)/n_t$.

Part (i) of Theorem 1 can be strengthened in the case $J = \emptyset$. By Theorem 1, if (1) forms a 1-cover, then $\sum_{s \in I} 1/n_s \in \mathbb{Z}$ for some nonempty subset I of $\{1, \dots, k\}$, which is the main result of M. Z. Zhang [Z] obtained by means of the Riemann zeta function. For an exact m -cover (1), the author proved in [S1] that for each $n = 0, 1, \dots, m$ there exist at least $\binom{m}{n}$ subsets I of $\{1, \dots, k\}$ with $\sum_{s \in I} 1/n_s = n$. When (1) is an m -cover and m_1, \dots, m_k are positive integers, it was shown in [S3] that there are at least m positive integers in the form $\sum_{s \in I} m_s/n_s$ where $I \subseteq \{1, \dots, k\}$; we even conjecture that there exist nonempty subsets I_1, \dots, I_m of $\{1, \dots, k\}$ for which $I_1 \subset \dots \subset I_m$ and $\sum_{s \in I_t} m_s/n_s \in \mathbb{Z}$ for all $t = 1, \dots, m$.

The first part of Theorem 1 yields

Corollary 1. *Let (1) be an m -cover of \mathbb{Z} and m_1, \dots, m_k any integers. Then*

$$(6) \quad \left| \left\{ \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} : I \subseteq \{1, \dots, k\} \right\} \right| \leq \frac{2^k}{m+1}.$$

Proof. By part (i) of Theorem 1, for any $J \subseteq \{1, \dots, k\}$ there are at least $m + 1$ subsets I of $\{1, \dots, k\}$ with $\{\sum_{s \in I} m_s/n_s\} = \{\sum_{s \in J} m_s/n_s\}$. Since $\{1, \dots, k\}$ has exactly 2^k subsets, Corollary 1 follows immediately. \square

Remark 1. A conjecture of P. Erdős proved by R. B. Crittenden and C. L. Vanden Eynden [CV] states that (1) forms a 1-cover of \mathbb{Z} if it covers $1, \dots, 2^k$. In [S2], [S3] the author showed that (1) forms an m -cover of \mathbb{Z} if there exist W consecutive integers each of which lies in at least m members of (1), where W is the least integer equal to the left hand side of (6) for some integers m_1, \dots, m_k prime to n_1, \dots, n_k respectively.

As for part (ii) of Theorem 1 we should mention the following result obtained by the author ([S4]) recently: Let (1) be an exact m -cover of \mathbb{Z} , and J a nonempty subset of $\{1, \dots, k\}$ with $(n_s, n_t) \mid a_s - a_t$ for all $s, t \in J$ (i.e. $\emptyset \neq J \subseteq S(x)$ for some $x \in \mathbb{Z}$). Then

$$\left| \left\{ I \subseteq J^- : \left\{ \sum_{s \in I} \frac{1}{n_s} \right\} = \frac{a}{N(J)} \right\} \right| \geq \frac{\prod_{s \in J} n_s}{N(J)}$$

for every $a = 0, 1, \dots, N(J) - 1$, and

$$\left| \left\{ I \subseteq J^- : \sum_{s \in I} \frac{1}{n_s} = \frac{a}{N(J)} \right\} \right| \geq \binom{m-1}{[a/N(J)]}$$

for all $a = 0, 1, 2, \dots$ if $|J| = 1$.

Corollary 2. *Let (1) be an m -cover of \mathbb{Z} with $n_1 \leq \dots \leq n_{k-1} \leq n_k$. Suppose that $B = \{a_s(n_s)\}_{s=1}^{k-1}$ fails to be an m -cover of \mathbb{Z} . If $\sum_{s=1}^{k-1} 1/n_s = m$, then $n_{k-1} = n_k > 1$ and*

$$(7) \quad \left\{ \sum_{s \in I} \frac{1}{n_s} : I \subseteq \{1, \dots, k-1\} \right\} \supseteq \left\{ \frac{r}{n_k} : r = 0, 1, \dots, n_k - 1 \right\}.$$

Proof. Assume that $\sum_{s=1}^{k-1} 1/n_s = m$. By part (ii) of Theorem 1 there exists an $\alpha \in [0, 1)$ such that

$$\begin{aligned} & \left\{ \left\{ \sum_{s \in I} \frac{1}{n_s} \right\} : I \subseteq \{1, \dots, k-1\}, \left[\sum_{s \in I} \frac{1}{n_s} \right] \geq m-1 \right\} \\ & \supseteq \left\{ \frac{a}{n_k} : 0 \leq a < n_k, \{a\} = \alpha \right\}. \end{aligned}$$

Thus

$$\begin{aligned} & \left\{ \sum_{s \in J} \frac{1}{n_s} : J \subseteq \{1, \dots, k-1\}, \sum_{s \in J} \frac{1}{n_s} \notin \mathbb{Z} \right\} \\ & \supseteq \left\{ \sum_{s=1}^{k-1} \frac{1}{n_s} - \sum_{s \in I} \frac{1}{n_s} : I \subseteq \{1, \dots, k-1\}, m-1 < \sum_{s \in I} \frac{1}{n_s} < m = \sum_{s=1}^{k-1} \frac{1}{n_s} \right\} \\ & = \left\{ 1 - \left\{ \sum_{s \in I} \frac{1}{n_s} \right\} : I \subseteq \{1, \dots, k-1\}, \left[\sum_{s \in I} \frac{1}{n_s} \right] \geq m-1 \right\} \setminus \{1\} \\ & \supseteq \left\{ 1 - \frac{a}{n_k} : 0 \leq a < n_k, \{a\} = \alpha \right\} \setminus \{1\} = \left\{ \frac{b}{n_k} : 0 < b < n_k, \{b\} = \{-\alpha\} \right\}. \end{aligned}$$

Observe that (7) follows if $\alpha = 0$. Since B doesn't form an m -cover of \mathbb{Z} , we cannot have $n_1 = \dots = n_{k-1} = 1$ (otherwise $k-1 = \sum_{s=1}^{k-1} 1/n_s = m$). So $n_k \geq n_{k-1} > 1$;

hence by the above for some nonempty $J \subseteq \{1, \dots, k-1\}$ we have

$$\frac{1}{n_{k-1}} \leq \min_{s \in J} \frac{1}{n_s} \leq \sum_{s \in J} \frac{1}{n_s} = \frac{1-\alpha}{n_k} \leq \frac{1}{n_k} \leq \frac{1}{n_{k-1}}.$$

Therefore $n_k = n_{k-1}$ and $\alpha = 0$. We are done. □

Remark 2. Let (1) be an m -cover of \mathbb{Z} with $n_1 \leq \dots \leq n_{k-1} < n_k$. By part (iv) of Theorem I of [S3], $\sum_{s=1}^{k-1} 1/n_s \geq m$. In view of Corollary 2, if $\{a_s(n_s)\}_{s=1}^{k-1}$ fails to be an m -cover of \mathbb{Z} , then $\sum_{s=1}^{k-1} 1/n_s$ must be greater than m . This extends and improves a confirmed conjecture of Erdős which states that $\sum_{s=1}^k 1/n_s > 1$ for any 1-cover (1) with $1 < n_1 < \dots < n_{k-1} < n_k$ (see [E] and [G]).

Corollary 3. *Let (1) be an m -cover of \mathbb{Z} , and J a nonempty subset of $\{1, \dots, k\}$ with $|\{s \in J^- : x \in a_s(n_s)\}| = m - |J|$ for some $x \in \mathbb{Z}$. Let $\varepsilon_s \in \{1, -1\}$ for those $s \in J^-$. Then*

$$(8) \quad \left| \left\{ \left\{ \sum_{s \in I} \frac{\varepsilon_s}{n_s} \right\} : I \subseteq J^- \right\} \right| \geq N(J).$$

Proof. This follows immediately from the second part of Theorem 1. □

Remark 3. With the help of a local-global result proved in [S2], in 1994 the author found Corollary 3 in the case $|J| = 1$ (see Section 3 of [S3]).

Corollary 4. *Let (1) be a minimal m -cover of \mathbb{Z} , and m_1, \dots, m_k any positive integers prime to n_1, \dots, n_k respectively. Then for every $t = 1, \dots, k$ all the numbers $0, 1/n_t, \dots, (n_t - 1)/n_t$ lie in the set*

$$(9) \quad \left\{ \left\{ \sum_{s \in I} \frac{m_s}{n_s} - \sum_{s \in J} \frac{m_s}{n_s} \right\} : I, J \subseteq \{1, \dots, k\} \setminus \{t\} \ \& \ \sum_{s \in I} \frac{m_s}{n_s}, \sum_{s \in J} \frac{m_s}{n_s} \geq m - 1 \right\}.$$

Proof. By part (ii) of Theorem 1 there is an $\alpha_t \in [0, 1)$ such that

$$\left\{ \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} : I \subseteq \{1, \dots, k\} \setminus \{t\}, \left[\sum_{s \in I} \frac{m_s}{n_s} \right] \geq m - 1 \right\}$$

contains $S_t = \{a/n_t : 0 \leq a < n_t, \{a\} = \alpha_t\}$. As $r/n_t = (\alpha_t + r)/n_t - \alpha_t/n_t$ for each $r = 0, 1, \dots, n_t - 1$, the desired result follows. □

Remark 4. In [S3] the author was able to prove Corollary 4 with $\sum_{s \in J} m_s/n_s \geq m - 1$ in (9) replaced by $\sum_{s \in J} m_s/n_s \geq m - 2$.

3. PROOF OF THEOREM 1

Let's recall a key result given by the author in [S2].

Proposition 1. *Let $\mathcal{A} = \{\alpha_s + \beta_s \mathbb{Z}\}_{s=1}^k$ where $\alpha_1, \dots, \alpha_k$ are real numbers and β_1, \dots, β_k are positive reals. Let m be a positive integer. Then \mathcal{A} forms an m -cover of \mathbb{Z} (i.e. $|\{1 \leq s \leq k : (x - \alpha_s)/\beta_s \in \mathbb{Z}\}| \geq m$ for all $x \in \mathbb{Z}$) if and only if*

$$(10) \quad \sum_{\substack{I \subseteq \{1, \dots, k\} \\ \{\sum_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} \binom{[\sum_{s \in I} 1/\beta_s]}{n} e^{2\pi i \sum_{s \in I} \alpha_s/\beta_s} = 0$$

holds for all $\theta \in [0, 1)$ and $n = 0, 1, \dots, m - 1$.

Lemma 1. *Let k, m, n be positive integers with $k > m - n \geq 0$. Then (1) forms an m -cover of \mathbb{Z} if and only if for each $I \subseteq \{1, \dots, k\}$ with $|I| = m - n$ system $A_I = \{a_s(n_s)\}_{s \in I^-}$ forms an n -cover of \mathbb{Z} .*

Proof. If (1) is an m -cover of \mathbb{Z} and I is a subset of $\{1, \dots, k\}$ with $|I| = m - n$, then for any integer x we have

$$|\{s \in I^- : x \equiv a_s \pmod{n_s}\}| \geq m - |I| = n;$$

therefore A_I is an n -cover of \mathbb{Z} .

Now suppose that A_I forms an n -cover of \mathbb{Z} for all $I \subseteq \{1, \dots, k\}$ with $|I| = m - n$. Let's show that $A = A_\emptyset$ forms an m -cover of \mathbb{Z} . Assume on the contrary that for some integer x set $J = \{1 \leq s \leq k : x \equiv a_s \pmod{n_s}\}$ has cardinality $l < m$. Choose a subset I of $\{1, \dots, k\}$ with cardinality $m - n$ such that either $I \subseteq J$ or $I \supseteq J$. Observe that x belongs to less than n members of A_I . This contradiction ends our proof. \square

Remark 5. Apparently for (1) to be an m -cover of \mathbb{Z} it is necessary that $k \geq m$.

Proof of part (i) of Theorem 1. It suffices to handle the case $m = m(A) > 0$.

At first we assume that n_1, \dots, n_k are all greater than one. Since $m \leq \sum_{s=1}^k 1/n_s \leq k/2$, either J or J^- has cardinality not less than m .

Case 1. $|J^-| \geq m$. Among $I \subseteq \{1, \dots, k\}$ with $\{\sum_{s \in I} m_s/n_s\} = \{\sum_{s \in J} m_s/n_s\}$, we select a J_0 with the least cardinality. Apparently $|J_0^-| \geq |J^-| \geq m$. Let $I_0 = \{s_1, \dots, s_{m-1}\}$ be a subset of $\{1, \dots, k\}$ with $|I_0| = m - 1$ and $I_0 \cap J_0 = \emptyset$. By Lemma 1 and Remark 5, system $\{a_s(n_s)\}_{s \in I_0^-}$ forms a 1-cover of \mathbb{Z} and hence so does $\mathcal{A}_0 = \{a_s + (n_s/m_s)\mathbb{Z}\}_{s \in I_0^-}$. As $J_0 \subseteq I_0^-$, by Proposition 1 or Theorem 2 of [S2] there is a $J_1 \subseteq I_0^-$ for which $J_1 \neq J_0$ and $\{\sum_{s \in J_1} m_s/n_s\} = \{\sum_{s \in J_0} m_s/n_s\}$. According to the choice of J_0 we must have $J_1 \not\subseteq J_0$. Choose $t_1 \in J_1 \setminus J_0$ and put $I_1 = \{t_1, s_2, \dots, s_{m-1}\}$. Observe that $I_1 \cap J_0 = \emptyset$. Since $\mathcal{A}_1 = \{a_s + (n_s/m_s)\mathbb{Z}\}_{s \in I_1^-}$ forms a 1-cover of \mathbb{Z} , there exists a $J_2 \subseteq I_1^-$ with $J_2 \neq J_0$ such that $\{\sum_{s \in J_2} m_s/n_s\} = \{\sum_{s \in J_0} m_s/n_s\}$. Choose $t_2 \in J_2 \setminus J_0$ and put $I_2 = \{t_1, t_2, s_3, \dots, s_{m-1}\}$. Then continue this procedure to find $J_3, t_3, I_3; \dots; J_{m-1}, t_{m-1}, I_{m-1}; J_m, t_m$ in the same way. Apparently J_1, J_2, \dots, J_m are all different from J_0 . If $1 \leq i < j \leq m$, then $t_i \in J_i \setminus J_j$ because $t_i \in I_{j-1}$ and $J_j \cap I_{j-1} = \emptyset$. So the $m + 1$ subsets $J_0, J_1, J_2, \dots, J_m$ of $\{1, \dots, k\}$ are distinct; therefore

$$\begin{aligned} & \left| \left\{ I \subseteq \{1, \dots, k\} : I \neq J \ \& \ \sum_{s \in I} \frac{m_s}{n_s} - \sum_{s \in J} \frac{m_s}{n_s} \in \mathbb{Z} \right\} \right| \\ & \geq |\{J_i : 0 \leq i \leq m \ \& \ J_i \neq J\}| \geq m. \end{aligned}$$

Case 2. $|J| \geq m$, i.e. $|(J^-)^-| \geq m$. It follows from the above that

$$\left| \left\{ I \subseteq \{1, \dots, k\} : I \neq J^- \ \& \ \sum_{s \in I} \frac{m_s}{n_s} - \sum_{s \in J^-} \frac{m_s}{n_s} \in \mathbb{Z} \right\} \right| \geq m.$$

Thus

$$\begin{aligned} & \left| \left\{ I' \subseteq \{1, \dots, k\} : I' \neq J \ \& \ \sum_{s \in I'} \frac{m_s}{n_s} - \sum_{s \in J} \frac{m_s}{n_s} \in \mathbb{Z} \right\} \right| \\ &= \left| \left\{ I^- : I \subseteq \{1, \dots, k\}, I^- \neq J \ \& \ \sum_{s \in I^-} \frac{m_s}{n_s} - \sum_{s \in J} \frac{m_s}{n_s} \in \mathbb{Z} \right\} \right| \\ &= \left| \left\{ I \subseteq \{1, \dots, k\} : I \neq J^- \ \& \ \sum_{s \in I} \frac{m_s}{n_s} - \sum_{s \in J^-} \frac{m_s}{n_s} \in \mathbb{Z} \right\} \right| \geq m. \end{aligned}$$

So far we have proven (4) in both cases.

Next let's consider the situation in which $K = \{1 \leq s \leq k : n_s = 1\}$ is nonempty. If $|K| < m$, then $\{a_s(n_s)\}_{s \in K^-}$ forms an $m - |K|$ -cover of \mathbb{Z} with all the moduli greater than one; hence by the above

$$\left| \left\{ I \subseteq K^- : \sum_{s \in I} \frac{m_s}{n_s} - \sum_{s \in J \setminus K} \frac{m_s}{n_s} \in \mathbb{Z} \right\} \right| \geq m - |K| + 1.$$

Therefore

$$\begin{aligned} & \left| \left\{ I \subseteq \{1, \dots, k\} : \sum_{s \in I} \frac{m_s}{n_s} - \sum_{s \in J} \frac{m_s}{n_s} \in \mathbb{Z} \right\} \right| \\ &= \left| \left\{ I \cup I' : I \subseteq K, I' \subseteq K^- \ \& \ \sum_{s \in I'} \frac{m_s}{n_s} - \sum_{s \in J \setminus K} \frac{m_s}{n_s} \in \mathbb{Z} \right\} \right| \\ &\geq \left| \left\{ I \cup I' : I \subseteq K, |I| \leq 1, I' \subseteq K^- \ \& \ \sum_{s \in I'} \frac{m_s}{n_s} - \sum_{s \in J \setminus K} \frac{m_s}{n_s} \in \mathbb{Z} \right\} \right| \\ &\geq |K| + \left| \left\{ I' \subseteq K^- : \sum_{s \in I'} \frac{m_s}{n_s} - \sum_{s \in J \setminus K} \frac{m_s}{n_s} \in \mathbb{Z} \right\} \right| \\ &\geq |K| + \max\{m - |K| + 1, 1\} \geq m + 1. \end{aligned}$$

This completes the proof. □

Lemma 2. *Let (1) be a system of arithmetic sequences, and J a nonempty subset of $\{1, \dots, k\}$ with $|J| \leq m(A)$ and $\bigcap_{s \in J} a_s(n_s) \neq \emptyset$. For each $s \in J^-$ let m_s be a positive integer. Let $0 \leq a < N(J)$ and*

$$(11) \quad C(a) = \sum_{\substack{I \subseteq J^- \\ \{\sum_{s \in I} \frac{m_s}{n_s}\} = \frac{a}{N(J)}}} (-1)^{|I|} \binom{[\sum_{s \in I} \frac{m_s}{n_s}]}{m(A) - |J|} e^{2\pi i \sum_{s \in I} \frac{m_s}{n_s} (a_s - a_J)}$$

where a_J is the unique integer in $\bigcap_{s \in J} a_s(n_s)$ with $0 \leq a_J < N(J)$. Then $C(a) = C(\{a\})$.

Proof. Apparently it suffices to show $C(a) = C(a - 1)$ providing $a \geq 1$.

Let $m = m(A)$. Observe that the sequences $a_s + (n_s/m_s)\mathbb{Z}$ ($s \in J^-$) together with $a_J + N(J)\mathbb{Z}$ form an $m - |J| + 1$ -cover of \mathbb{Z} . In view of Proposition 1,

$$\begin{aligned} & \sum_{\substack{I \subseteq J^- \\ \{\sum_{s \in I} \frac{m_s}{n_s}\} = \frac{a}{N(J)}}} (-1)^{|I|} \binom{\lfloor \sum_{s \in I} \frac{m_s}{n_s} \rfloor}{m - |J|} e^{2\pi i \sum_{s \in I} \frac{a_s m_s}{n_s}} \\ + & \sum_{\substack{I \subseteq J^- \\ \{\sum_{s \in I} \frac{m_s}{n_s} + \frac{1}{N(J)}\} = \frac{a}{N(J)}}} (-1)^{|I|+1} \binom{\lfloor \sum_{s \in I} \frac{m_s}{n_s} + \frac{1}{N(J)} \rfloor}{m - |J|} e^{2\pi i (\sum_{s \in I} \frac{a_s m_s}{n_s} + \frac{a_J}{N(J)})} \end{aligned}$$

vanishes. So

$$\begin{aligned} & e^{2\pi i a a_J / N(J)} C(a) \\ = & \sum_{\substack{I \subseteq J^- \\ \{\sum_{s \in I} \frac{m_s}{n_s}\} = \frac{a}{N(J)}}} (-1)^{|I|} \binom{\lfloor \sum_{s \in I} \frac{m_s}{n_s} \rfloor}{m - |J|} e^{2\pi i \sum_{s \in I} \frac{a_s m_s}{n_s}} \\ = & \sum_{\substack{I \subseteq J^- \\ \{\sum_{s \in I} \frac{m_s}{n_s}\} = \frac{a-1}{N(J)}}} (-1)^{|I|} \binom{\lfloor \sum_{s \in I} \frac{m_s}{n_s} \rfloor + \frac{a}{N(J)}}{m - |J|} e^{2\pi i (\sum_{s \in I} \frac{a_s m_s}{n_s} + \frac{a_J}{N(J)})} \\ = & e^{2\pi i a_J / N(J)} \sum_{\substack{I \subseteq J^- \\ \{\sum_{s \in I} \frac{m_s}{n_s}\} = \frac{a-1}{N(J)}}} (-1)^{|I|} \binom{\lfloor \sum_{s \in I} \frac{m_s}{n_s} \rfloor}{m - |J|} e^{2\pi i \sum_{s \in I} \frac{a_s m_s}{n_s}} \\ = & e^{2\pi i a_J / N(J)} e^{2\pi i (a-1) a_J / N(J)} C(a-1) = e^{2\pi i a a_J / N(J)} C(a-1). \end{aligned}$$

Therefore $C(a) = C(a - 1)$. We are done. □

Remark 6. If we replace $m(A) - |J|$ in (11) by a smaller nonnegative integer n , then the new $C(a)$ will equal zero by Proposition 1, because system $\{a_s + (n_s/m_s)\mathbb{Z}\}_{s \in J^-}$ forms an $m(A) - |J|$ -cover of \mathbb{Z} .

Proof of part (ii) of Theorem 1. Since $|\{s \in J^- : x \in a_s(n_s)\}| = m(A) - |J|$ for some integer x and $(m_s, n_s) = 1$ for all $s \in J^-$, system $\{a_s + (n_s/m_s)\mathbb{Z}\}_{s \in J^-}$ fails to form an $m(A) - |J| + 1$ -cover of \mathbb{Z} as well as $\{a_s(n_s)\}_{s \in J^-}$. As $x \in \bigcap_{s \in J^-} a_s(n_s)$, there is a unique integer a_J with $0 \leq a_J < N(J)$ such that $a_J \equiv a_s \pmod{n_s}$ for all $s \in J$. By Proposition 1 and Remark 6 there exists a $\theta \in [0, 1)$ such that

$$C(N(J)\theta) e^{2\pi i a_J \theta} = \sum_{\substack{I \subseteq J^- \\ \{\sum_{s \in I} \frac{m_s}{n_s}\} = \theta}} (-1)^{|I|} \binom{\lfloor \sum_{s \in I} \frac{m_s}{n_s} \rfloor}{m(A) - |J|} e^{2\pi i \sum_{s \in I} \frac{a_s m_s}{n_s}} \neq 0.$$

Put $\alpha = \{N(J)\theta\}$. If $0 \leq a < N(J)$ and $\{a\} = \alpha$, then $a - N(J)\theta \in \mathbb{Z}$ and hence $C(a) = C(N(J)\theta) \neq 0$ by Lemma 2; therefore $\{\sum_{s \in I} \frac{m_s}{n_s}\} = \frac{a}{N(J)}$ for some $I \subseteq J^-$ with $\lfloor \sum_{s \in I} \frac{m_s}{n_s} \rfloor \geq m(A) - |J|$. This concludes the proof. □

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