PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 5, Pages 1413–1426
S 0002-9939(99)05049-2
Article electronically published on January 29, 1999

ESTIMATES FOR THE GREEN FUNCTION OF A GENERAL STURM-LIOUVILLE OPERATOR AND THEIR APPLICATIONS

N. CHERNYAVSKAYA AND L. SHUSTER

(Communicated by Hal L. Smith)

Abstract. For a general Sturm-Liouville operator with nonnegative coefficients, we obtain two-sided estimates for the Green function, sharp by order on the diagonal.

1. Introduction

In this paper we study the equation:

$$-(r(x)y'(x))' + q(x)y(x) = f(x), \quad x \in R. \tag{1.1}$$

Here and throughout the paper, $f(x) \in L_p(R)$, $p \in [1,\infty)$, and $r(x)$ and $q(x)$ satisfy the following conditions:

$$r(x) > 0,\ q(x) \geq 0,\ x \in R\ ; \quad \frac{1}{r(x)} \in L_1^{\text{loc}}(R),\ q(x) \in L_1^{\text{loc}}(R). \tag{1.2}$$

$$\lim_{|d|\to\infty} \int_{x-d}^{x} \frac{dt}{r(t)} \cdot \int_{x-d}^{x} q(t)\ dt = \infty, \qquad x \in R. \tag{1.3}$$

Conditions (1.2)-(1.3) guarantee that (1.1) has a unique solution in $L_p(R)$ for $p \in [1,\infty)$ (see §2), and under some additional requirement to $r(x)$ and $q(x)$ ([5]; see also §2 and (1.10) below), the two following assertions hold simultaneously:

1) For any $f(x) \in L_p(R)$, equation (1.1) has the unique solution $y(x) \in L_p(R)$:

$$y(x) = (Gf)\ (x) \overset{\text{def}}{=} \int_{-\infty}^{\infty} G(x,t)\ f(t)\ dt, \quad x \in R, \tag{1.4}$$

2)

$$\|G\|_{p\to p} < \infty. \tag{1.5}$$

Here (see §2), $G(x,t)$ is the Green function corresponding to (1.1)

$$G(x,t) = \begin{cases} u(x)v(t) & \text{if} \quad x \geq t, \\ u(t)v(x) & \text{if} \quad x \leq t, \end{cases} \tag{1.6}$$

Received by the editors August 21, 1997.
1991 *Mathematics Subject Classification.* Primary 34B27.
Research of the first author supported by the Israel Academy of Sciences, under Grant 431/95.
Research of the second author supported by the Israel Academy of Sciences, under Grant 505/95.

and $\{u(x), v(x)\}$ is a special fundamental system of solutions (FSS) of the equation

$$(r(x)z'(x))' = q(x)z(x)\ , \quad x \in R. \tag{1.7}$$

Below, $G(x,t)$ stands for the Green function of a Sturm-Liouville operator $\mathcal{L}$. Here, assuming 1)-2), $\mathcal{L}$ denotes a differential expression $\ell y = -(r(x)y'(x))' + q(x)y(x)$ defined by the solutions of (1.1), while $f(x)$ changes in $L_p(R)$ [3]. The goal of the paper is to obtain two-sided estimates for $G(x,t)$. We get such inequalities following the methods suggested in [2], [3]. (In [2], [3] we considered the case $r(x) \equiv 1,\ 1 \le q(x) \in L_1^{\mathrm{loc}}(R)$). In particular, as in [2], [3], we rely on a special formula which gives a representation of $G(x,t)$ via its diagonal values:

$$G(x,t) = \sqrt{\rho(x)\rho(t)} \exp\left(-\frac{1}{2}\left|\int_x^t \frac{d\xi}{r(\xi)\rho(\xi)}\right|\right),\ \rho(x) = G(x,t)\ \Big|_{t=x}\ . \tag{1.8}$$

(In [6], formula (1.8) has been obtained for $r(x) \equiv 1$). The main advantage of (1.8) compared with (1.6) is the following: under conditions (1.2)-(1.3), in contrast to $\{u(x), v(x)\}$, there have been obtained a priori two-sided estimates for $\rho(x)$ which are sharp by order ([4], [5]; see Theorem 2.2 below):

$$2^{-1}\left(\int_{x-d_1(x)}^{x+d_2(x)} q(\xi)d\xi\right)^{-1} \le \rho(x) \le 2\left(\int_{x-d_1(x)}^{x+d_2(x)} q(\xi)d\xi\right)^{-1}, \quad x \in R. \tag{1.9}$$

Here, $d_{1,2}(x)$ are auxiliary functions in $r(x), q(x)$ (see §2). Formulae (1.8) and (1.9) immediately lead to estimates of $G(x,t)$ for $x \ne t$ (Theorem 2.2). The obtained inequalities for $G(x,t)$ proves useful for the study of (1.1). As an example of their application, we consider a problem of validity of 1)-2). We show that 1)-2) hold if

$$\mathcal{A} \overset{\text{def}}{=} \inf_{x \in R}\left(\frac{1}{2d(x)}\int_{x-d(x)}^{x+d(x)} q(\xi)d\xi\right) > 0. \tag{1.10}$$

([5]; see §2, Theorem 2.3 below). Here, $d(x)$ is an auxiliary continuous function, positive for $x \in R$, which can be constructed from $r(x)$ and $q(x)$ (§2). In other words, 1)-2) hold if some special average of $q(x)$ (of Steklov type [11]) is separated from zero on the whole number axis. Perhaps condition (1.10) deserves special attention because it is valid for equation (1.1) with oscillating coefficients $q(x)$ (see the example in §2). To conclude, note that Theorem 2.2 proves useful in many other problems related to (1.1). For example, using it one can obtain necessary and sufficient conditions:

(1) for the operator $G\colon L_p(R) \to L_p(R)$ to be compact;
(2) for the operator $\mathcal{L}$ to be coercive;
(3) for solvability of the Neumann and Dirichlet problems for equation (1.1).

These results will be presented in our forthcoming papers.

2. Statement of results. Example

Throughout we denote by τ absolute positive constants, not essential for exposition, which may differ within a single chain of calculations.

Theorem 2.1. *Suppose that one has* (1.2) *and* (2.1):

$$\int_{-\infty}^0 q(\xi)d\xi > 0, \quad \int_0^\infty q(\xi)d\xi > 0. \tag{2.1}$$

Then there exists a FSS of (1.7) $\{u(x), v(x)\}$ *such that*

$$\begin{aligned} &u(x) > 0, \quad v(x) > 0, \quad u'(x) \le 0, \quad v'(x) \ge 0 \quad \text{for} \quad x \in R, \\ &r(x)\ [v'(x)u(x) - u'(x)v(x)] = 1 \qquad \text{for} \quad x \in R, \\ &\lim_{x\to-\infty} \frac{v(x)}{u(x)} = \lim_{x\to\infty} \frac{u(x)}{v(x)} = 0. \end{aligned} \tag{2.2}$$

Corollary 2.1.1. *Suppose that one has* (1.2) *and* (2.1). *Then* (1.7) *has no solutions* $z(x) \in L_p(R)$ *apart from* $z(x) \equiv 0$.

Definition 2.1. An FSS of (1.6) satisfying (2.2) is called a principal FSS (PFSS).

Lemma 2.1. *For* $\{u(x), v(x)\}$ *one has the following representations:*

$$\begin{aligned} u(x) &= \sqrt{\rho(x)} \exp\left(-\frac{1}{2}\int_{x_1}^{x} \frac{d\xi}{r(\xi)\rho(\xi)}\right), \\ v(x) &= \sqrt{\rho(x)} \exp\left(\frac{1}{2}\int_{x_1}^{x} \frac{d\xi}{r(\xi)\rho(\xi)}\right) \end{aligned} \tag{2.3}$$

where $x \in R$, $\rho(x) = u(x)v(x)$, x_1 *is the unique solution of the equation* $u(x) = v(x)$ *in* R. *Moreover, for* $G(x,t)$ *and* $\rho(x)$ *one has representation* (1.8) *and* (2.4):

$$\int_{-\infty}^{0} \frac{dt}{r(t)\rho(t)} = \int_{0}^{\infty} \frac{dt}{r(t)\rho(t)} = \infty. \tag{2.4}$$

Formulae (2.3) were applied in [13, §19.53], [4], [5], and, for $r(x) \equiv 1$, in [2], [3], [6].

Lemma 2.2. *For every fixed* $x \in R$, *each of the equations in* $d \ge 0$:

$$1 = \int_{x-d}^{x} \frac{dt}{r(t)} \cdot \int_{x-d}^{x} q(t)dt, \quad 1 = \int_{x}^{x+d} \frac{dt}{r(t)} \cdot \int_{x}^{x+d} q(t)dt \tag{2.5}$$

has a unique finite positive solution.

Denote the solutions of (2.5) by $d_1(x), d_2(x)$, respectively. For $x \in R$ let us introduce the following functions:

$$\begin{aligned} \varphi(x) &= \int_{x-d_1(x)}^{x} \frac{dt}{r(t)}, \quad \psi(x) = \int_{x}^{x+d_2(x)} \frac{dt}{r(t)}, \\ h(x) &= \frac{\varphi(x)\psi(x)}{\varphi(x)+\psi(x)} \equiv \left(\int_{x-d_1(x)}^{x+d_2(x)} q(\xi)d\xi\right)^{-1}. \end{aligned} \tag{2.6}$$

Theorem 2.2. *For* $x, t \in R$ *one has inequalities:*

$$\begin{aligned} 2^{-1}\ v(x) &\le (r(x)v'(x))\varphi(x) \le 2v(x), \\ 2^{-1}u(x) &\le (r(x)|u'(x)|)\psi(x) \le 2u(x), \end{aligned} \tag{2.7}$$

$$2^{-1}\ h(x) \le \rho(x) \le 2h(x), \tag{2.8}$$

$$2^{-1}\ \exp\left(-\left|\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right|\right) \le \frac{G(x,t)}{\sqrt{h(x)h(t)}} \le 2\ \exp\left(-\frac{1}{4}\left|\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right|\right). \tag{2.9}$$

Corollary 2.2.1. *Suppose that one has* (1.2)-(1.3) *and* (2.10):

$$a^{-1} \le r(x) \le a \quad \text{for} \quad x \in R, \quad a = \text{const}, \quad a \in [1, \infty). \tag{2.10}$$

Consider the following equation in $d \ge 0$ for a fixed $x \in R$:

$$S(d) = 2, \qquad S(d) \overset{\text{def}}{=} d \int_{x-d}^{x+d} q(\xi)\, d\xi. \tag{2.11}$$

Equation (2.11) *has a unique finite positive solution for every $x \in R$. If $\tilde{d}(x)$ is the solution of* (2.11), *then the following estimates hold:*

$$(2a+2)^{-1}\tilde{d}(x) \le \rho(x) \le 2^{-1}(2a+1)\ \tilde{d}(x), \qquad x \in R. \tag{2.12}$$

Remark. The function $\tilde{d}(x)$ has been introduced by M. Otelbaev ([7]). Estimates (2.7)-(2.8) are given in [4], [5].

For $q(x) \ge \varepsilon > 0$, $x \in R$, estimates (2.7)-(2.8) with other, more complicated auxiliary functions were obtained in [8]. The method in [8] does not allow us to remove the restriction $q(x) \ge \varepsilon > 0$ because it uses division by $q(x)$. Our method can be viewed as a development of [1], [3], [10]. See [5, §2] for a review of the results of type (2.7)-(2.8).

Lemma 2.3. *For every fixed $x \in R$, the equation in $d \ge 0$:*

$$1 = \int_{x-d}^{x+d} \frac{dt}{r(t)h(t)} \tag{2.13}$$

has a unique finite positive solution. Denote this solution by $d(x)$. The function $d(x)$ is continuous for $x \in R$.

Theorem 2.3. *If $\mathcal{A} > 0$ (see* (1.10)*), $p \in [1, \infty)$, then assertions* 1)-2)*, in* §1*, are valid and*

$$\left\| q(x)^{1/p} y(x) \right\|_p \le (\tau\mathcal{A})^{-1/p'} \left\| f \right\|_p. \tag{2.14}$$

Corollary 2.3.1. *Let $f(x) \in L_1(R)$. If $\mathcal{A} > 0$, then* (1.1) *has the unique solution $y(x)$ in $L_1(R)$,* (1.4)-(1.5) *hold, and, moreover,*

$$\|(r(x)y'(x))'\|_1 + \|q(x)y(x)\|_1 \le 3\ \|f(x)\|_1. \tag{2.15}$$

Remark. Corollary 2.3.1 generalizes the corresponding results of [8].

Usually one is not able to calculate exact values of $\varphi(x), \psi(x), d(x)$. However, to apply Theorems 2.2-2.3, it is sufficient to have two-sided estimates of these functions.

The following assertion gives a method for obtaining such estimates. This theorem is technical, and we give it without proof. See [4], [5] for details.

Theorem 2.4. *Suppose that* (1.2) *holds and there are continuous positive functions $r_1(x), q_1(x)$ and functions $r_2(x), q_2(x)$ such that*

1)

$$r(x) = r_1(x) + r_2(x), \quad q(x) = q_1(x) + q_2(x), \quad x \in R; \tag{2.16}$$

2) *there are constants $a \ge 1, b > 0$ such that for $|x| \gg 1$ one has the inequalities*

$$\frac{1}{a} \le \frac{r_1(t)}{r_1(x)},\ \frac{q_1(t)}{q_1(x)} \le a \qquad \text{for} \quad |t - x| \le b\hat{d}(x),\ \hat{d}(x) = \sqrt{\frac{r_1(x)}{q_1(x)}}; \tag{2.17}$$

3) *there is a constant* $\delta \in (0,1]$ *such that* $r(x) \ge \delta r_1(x)$ *for* $x \in R$;
4) $\varkappa_1(x) \to 0,\ \varkappa_2(x) \to 0$ *as* $|x| \to \infty$, *where*

$$\begin{aligned}\varkappa_1(x) &= \frac{1}{\sqrt{r_1(x)q_1(x)}} \sup_{|z|\le b\hat d(x)} \left|\int_x^{x+z} q_2(t)\ dt\right|,\\ \varkappa_2(x) &= \sqrt{r_1(x)q_1(x)} \sup_{|z|\le b\hat d(x)} \left|\int_x^{x+z} \frac{r_2(t)}{r_1(t)^2}\ dt\right|.\end{aligned} \tag{2.18}$$

Then the following assertions hold:

A) *If conditions* 1)-4) *are satisfied and* $b \ge 3a$, *then* (1.3) *holds. In addition,*

$$\tau^{-1}(r_1(x)q_1(x))^{-1/2} \le \varphi(x), \psi(x) \le \tau(r_1(x)q_1(x))^{-1/2},\quad |x| \gg 1, \tag{2.19}$$

$$\tau^{-1}(r_1(x)q_1(x))^{-1/2} \le h(x) \le \tau(r_1(x)q_1(x))^{-1/2},\qquad x \in R. \tag{2.20}$$

B) *Under the hypotheses of* A), *if* $r(x) \le \tau r_1(x)$, *then*

$$\begin{aligned}&\frac{\tau^{-1}}{\sqrt[4]{r_1(t)r_1(x)q_1(t)q_1(x)}}\ \exp\left(-\tau\left|\int_x^t \sqrt{\frac{q_1(\xi)}{r_1(\xi)}}\ d\xi\right|\right) \le G(x,t)\\ &\le \frac{\tau}{\sqrt[4]{r_1(t)r_1(x)q_1(t)q_1(x)}}\ \exp\left(-\frac{1}{\tau}\left|\int_x^t \sqrt{\frac{q_1(\xi)}{r_1(\xi)}}\ d\xi\right|\right),\quad x,t \in R.\end{aligned} \tag{2.21}$$

C) *If conditions* 1)-4) *are satisfied, and* $b \ge 160a^3\delta^{-2}$, *then (see* (2.17)*):*

$$\tau^{-1}\hat d(x) \le d(x) \le \tau \hat d(x),\qquad x \in R. \tag{2.22}$$

D) *If, in addition, the function* $q(x)$ *is such that for any* $\alpha > 0$ *and* $x \in R$ *one has the inequality* $\int_{x-\alpha}^{x+\alpha} q(\xi)d\xi > 0$, *then* $\mathcal{A} > 0$ *(see* (1.10)*) if and only if* $\inf_{x\in R} q_1(x) > 0$.

Example. Consider equation (1.1) with the coefficients for $x \in R$:

$$r(x) = \left(1+x^2\right)^{-1} + 2^{-1}\left(1+x^2\right)^{-1}\cos\left(e^{|x|}\right),\quad q(x) = 2e^{|x|}\cos^2\left(e^{|x|}\right). \tag{2.23}$$

According to Theorem 2.4, choose $r_1(x) = \left(1+x^2\right)^{-1}, q_1(x) = e^{|x|}$. Then (see (2.17)) $\hat d(x) = \left(1+x^2\right)^{-1/2} e^{-|x|/2} \to 0$ as $|x| \to \infty$. Let us verify that for any fixed $b > 0$ all the hypotheses of Theorem 2.3 are satisfied. One can assume $x \ge 0$ because all the above functions are even. Since $\hat d(x) \to 0$ as $x \to \infty$ and

$$\frac{r_1(x)}{r_1(t)} = 1 + (t-x)\ \frac{t+x}{1+x^2}\ ,\qquad \frac{q_1(t)}{q_1(x)} = e^{t-x}, \tag{2.24}$$

one can see that 2) holds, say for $a = 5/4$. Clearly in this case, $\delta = 1/2$ and 3) also holds. Further,

$$\varkappa_1(x) = \frac{\sqrt{1+x^2}}{e^{x/2}} \sup_{|z|\le b\hat d(x)} \left|\int_x^{x+z} e^t\cos(e^t)dt\right| \le \frac{2\sqrt{1+x^2}}{e^{x/2}}.$$

To estimate $\varkappa_2(x)$, we use the second mean theorem [12, Ch.12, §3] which can be applied here for $x \gg 1$:

$$\varkappa_2(x) = \frac{1}{2}\,\frac{e^{x/2}}{\sqrt{1+x^2}} \sup_{|z|\le b\hat{d}(x)} \left| \int_x^{x+z} \left[\frac{1+t^2}{e^t}\right] \left[e^t \cos e^t\right] dt \right|$$
$$\le \frac{\tau e^{x/2}}{\sqrt{1+x^2}} \sup_{|z|\le b\hat{d}(x)} \frac{1+x^2}{e^x} \left| \int_x^{x+z} e^t \cos e^t dt \right| \le \tau\,\frac{\sqrt{1+x^2}}{e^{x/2}}.$$

From the estimates for $\varkappa_{1,2}(x)$, it follows that all the hypotheses of Theorem 2.4, including A), B), C), are satisfied. Hence

$$\tau^{-1}\sqrt{1+x^2}\; e^{-|x|/2} \le \rho(x) \le \tau\sqrt{1+x^2}\; e^{-|x|/2}, \qquad x \in R,$$
$$G(x,t) \le \tau \sqrt[4]{(1+t^2)(1+x^2)}\; e^{-\frac{|x|+|t|}{4}}\; \exp\left(-\frac{1}{\tau}\left|\int_x^t \sqrt{(1+\xi^2)}e^{\xi/2}d\xi\right|\right),\ x,t\in R.$$

From the graph of $q(x)$ one can see that D) also holds, and $q_1(x) \ge 1$ for $x \in R$. Hence, by Theorem 2.3 for (1.1) with coefficients (2.23), one has assertions 1)-2) of §1.

Remark. From the proofs given below it follows that one can replace condition (1.3) with a weaker one:

$$2\mu \overset{\text{def}}{=} \inf_{x\in R} \lim_{|d|\to\infty} \left(\int_{x-d}^x \frac{dt}{r(t)} \cdot \int_{x-d}^x q(t)dt\right) > 0$$

without essentially affecting the main results of the paper. In particular, if $\mu < \infty$, then to obtain estimates for $\rho(x)$ and $G(t,x)$, one has to consider the equations

$$\mu = \int_{x-d}^x \frac{dt}{r(t)} \int_{x-d}^x q(t)dt, \qquad \mu = \int_x^{x+d} \frac{dt}{r(t)} \int_x^{x+d} q(t)dt$$

instead of (2.5), and then to repeat the proofs from §3 - §5.

3. Construction of a PFSS and its properties

In this section we prove Theorem 2.1. We need the following lemma.

Lemma 3.1. *Suppose that* (1.2) *and* (2.1) *hold. Then the solution $z(x)$ of the Cauchy problem for* (1.7) *with $z(0)=1,\ z'(0)=0$, satisfies the following relations:*

$$(3.1)\qquad z(x) \ge 1,\quad \text{sign }\ z'(x) = \text{sign }\ x \quad \textit{for}\quad x \in R, \quad c_0 \overset{\text{def}}{=} \int_{-\infty}^{\infty} \frac{d\xi}{r(\xi)z^2(\xi)} < \infty.$$

Proof. Let us verify that $z(x) > 0$ for $x \in R_+$. Since $z(0) = 1$, there is a neighborhood of zero where $z(x) > 0$. Suppose that $z(x)$ has zeroes on $x \in R_+$, and let $x_0 > 0$ be the first zero of $z(x)$. Then $z'(x_0) \le 0$. Indeed, otherwise $z(t) < 0$ for $t < x_0$, provided t is sufficiently close to x_0. But then there is a root of $z(x)$ on $(0, x_0)$, a contradiction to the choice of x_0. On the other hand,

$$r(x_0)\ z'(x_0) = \int_0^{x_0} q(\xi)\ z(\xi)\ d\xi \ge 0.$$

Hence $z(x_0) = z'(x_0) = 0$, i.e., $z(x) \equiv 0$ for $x \in R$, a contradiction to $z(0) = 1$. Thus, for $x \in R_+$ one has $z(x) > 0 \Rightarrow z'(x) \ge 0 \Rightarrow z(x) \ge 1,\ x \ge 0$. From (1.2) and

(2.1) it follows that there exists $\tau \gg 1$ such that $\int_0^\tau q(\xi)d\xi \geq \tau^{-1}$. By the above proof we get

$$r(\tau)z'(\tau) = \int_0^\tau q(\xi)z(\xi)d(\xi) \geq \int_0^\tau q(\xi)d\xi \geq \frac{1}{\tau}.$$

Since $r(x)z'(x)$ is a nondecreasing function, then $r(x)z'(x) \geq r(\tau)z'(\tau) \geq \tau^{-1}$ for $x \geq \tau$. Hence

$$z(x) \geq z(\tau) + \frac{1}{\tau}\int_\tau^x \frac{d\xi}{r(\xi)} \geq 1 + \frac{1}{\tau}\int_\tau^x \frac{d\xi}{r(\xi)}, \quad x \geq \tau.$$

From the latter estimate it follows that

$$\begin{aligned}\int_\tau^\infty \frac{d\xi}{r(\xi)z^2(\xi)} &\leq \int_\tau^\infty \frac{1}{r(\xi)}\left[1 + \frac{1}{\tau}\int_\tau^\xi \frac{ds}{r(s)}\right]^{-2} d\xi \\ &= \tau\left\{1 - \left[1 + \frac{1}{\tau}\int_\tau^\infty \frac{ds}{r(s)}\right]^{-1}\right\} \leq \tau.\end{aligned}$$

Clearly, $\big(r(x)z^2(x)\big)^{-1} \in L_1(0,\tau)$. Thus, $\big(r(x)z(x)^2\big)^{-1} \in L_1(R_+)$. One can similarly check the other statements of the lemma. $\square$

Proof of Theorem 2.1. Let $z(x)$ be the solution $z(x)$ from Lemma 3.1. Set

$$v(x) = \frac{z(x)}{\sqrt{c_0}}\int_{-\infty}^x \frac{d\xi}{r(\xi)z^2(\xi)}\ , \quad u(x) = \frac{z(x)}{\sqrt{c_0}}\int_x^\infty \frac{d\xi}{r(\xi)z^2(\xi)}\ , \quad x \in R \tag{3.2}$$

where c_0 is defined by (3.1). The functions $\{u(x), v(x)\}$, form a PFSS of (1.7). Let us establish (2.2). Verify that $u'(x) \leq 0$ for $x \in R$. Indeed, since by (3.1),

$$\sqrt{c_0}\ u'(x) = z'(x)\int_x^\infty \frac{d\xi}{r(\xi)z^2(\xi)} - \frac{1}{r(x)z(x)}\ , \qquad x \in R,$$

we have $u'(x) \leq 0$ for $x \leq 0$. Let $x > 0$. Since $r(x)z'(x)$ does not decrease, then

$$\begin{aligned}\sqrt{c_0}\ u'(x) &= z'(x)\int_x^\infty \frac{dt}{r(t)z^2(t)} - \frac{1}{r(x)z(x)} \leq \frac{1}{r(x)}\int_x^\infty \frac{z'(t)dt}{z^2(t)} - \frac{1}{r(x)z(x)} \\ &= -\frac{1}{r(x)z(\infty)} \leq 0.\end{aligned}$$

The remaining relations in (2.2) are either obvious or can be similarly checked. $\square$

Proof of Corollary 2.1.1. Assume the contrary. Then for some τ_1, τ_2 ($|\tau_1| + |\tau_2| \neq 0$), one has $z(x) = \tau_1 u(x) + \tau_2 v(x) \in L_p(R)$. For example, let $\tau_2 \neq 0$. By (2.2) there is $\tau > 0$ such that $u(x)v(x)^{-1}\left|\tau_1\tau_2^{-1}\right| \leq 2^{-1}$ for $x \geq \tau$. Then

$$\|z(x)\|_p^p > \int_\tau^\infty |\tau_1 u(t) + \tau_2 v(t)|^p\, dt \geq |\tau_2 v(\tau)|^p \int_\tau^\infty \left|1 - \left|\frac{\tau_1}{\tau_2}\ \frac{u(t)}{v(t)}\right|\right|^p dt = \infty.$$

Hence $\tau_2 = 0$. Since $u(x) \notin L_p(-\infty, 0)$, one also has $\tau_1 = 0$. $\square$

Remark. This method of construction of a PFSS was used in [9].

4. Estimates of the Green function

In this section we prove Theorem 2.2 and Corollary 2.2.1.

Proof of Lemma 2.1. Let $F(x) = v(x) - u(x)$. Then $F'(x) = v'(x) - u'(x) \geq 0$ for $x \in R$ (see (2.2)). From (3.2) it follows that $F(x) < 0$ as $x \to -\infty$, $F(x) > 0$ as $x \to \infty$. Then there is a unique root x_1 to $F(x) = 0$. Since

$$\frac{v'(t)}{v(t)} - \frac{u'(t)}{u(t)} = \frac{1}{r(t)u(t)v(t)} = \frac{1}{r(t)\rho(t)}, \quad t \in R \tag{4.1}$$

(see (2.2)), from (4.1) it follows that $\{u(x), v(x)\}$ are solutions to the system

$$\begin{cases} \dfrac{v(x)}{u(x)} = \exp\left(\displaystyle\int_{x_1}^{x} \frac{dt}{r(t)\rho(t)}\right) & x \in R, \\ u(x)v(x) = \rho(x) & x \in R. \end{cases} \tag{4.2}$$

Solving (4.2) with respect to $v(x), u(x)$ gives (2.3). To obtain (1.8), one has to substitute (2.3) into (1.6). □

Corollary 2.1.1. *To obtain* (2.4)*, one has to substitute* (2.3) *into* (2.2)*.*

Proof of Lemma 2.2. Clearly, $\Phi_1(0) = \Phi_2(0) = 0$, $\Phi_1(\infty) = \Phi_2(\infty) = \infty$, where

$$\Phi_1(d) = \int_{x-d}^{x} \frac{dt}{r(t)} \int_{x-d}^{x} q(t)dt, \quad \Phi_2(d) = \int_{x}^{x+d} \frac{dt}{r(t)} \int_{x}^{x+d} q(t)dt, \quad d \geq 0.$$

The functions $\Phi_{1,2}(d)$ do not decrease on $[0, \infty)$. Then the equations $\Phi_1(d) = 1$, $\Phi_2(d) = 1$ have unique finite positive solutions. □

Proof of Theorem 2.2. We use the method of [1], [3]. Let us check (2.7) for $v(x)$. (One can get the estimates for $u(x)$ similarly.) Let us integrate the equation $(r(\xi)v'(\xi))' = q(\xi)v(\xi)$, $\xi \in R$ along $[x-t, x]$, $t \geq 0$, divide the obtained integral by $r(x-t)$ and integrate the result by $t \in [0, s]$, $s \geq 0$; we finally obtain

$$\begin{aligned} r(x)v'(x) \int_{x-s}^{x} \frac{d\xi}{r(\xi)} &= v(x) - v(x-s) \\ &\quad + \int_{x-s}^{x} \frac{1}{r(t)} \int_{t}^{x} q(\xi)v(\xi)d\xi dt, \ x \in R, \ s \geq 0. \end{aligned} \tag{4.3}$$

In (4.3) set $s = d_1(x)$ (see (2.5)). Then from (2.2), (2.6) and (2.5) we obtain

$$r(x)v'(x)\varphi(x) < v(x) + v(x) \int_{x-d_1(x)}^{x} \frac{dt}{r(t)} \int_{x-d_1(x)}^{x} q(t)dt = 2v(x), \quad x \in R.$$

The right hand side of (2.7) is checked. Furthermore, from (4.3) it follows that

$$v(x) \leq v(x-s) + r(x)v'(x) \int_{x-s}^{x} \frac{d\xi}{r(\xi)}, \quad x \in R, \quad s \geq 0. \tag{4.4}$$

Let us multiply (4.4) by $q(x-s)$, integrate the result by $s \in [0, d_1(x)]$, and use (2.2), (2.6) and (2.5) again:

(4.5)
$$\begin{aligned}
&\frac{v(x)}{\varphi(x)} = v(x)\int_{x-d_1(x)}^{x} q(t)dt = v(x)\int_0^{d_1(x)} q(x-s)\, ds \\
&\le \int_0^{d_1(x)} q(x-s)v(x-s)ds + r(x)v'(x)\int_0^{d_1(x)} q(x-s)\int_{x-s}^{x}\frac{d\xi}{r(\xi)}\, ds \\
&\le \int_{x-d_1(x)}^{x} (r(\xi)v'(\xi))'\, d\xi + r(x)v'(x)\int_{x-d_1(x)}^{x} q(t)dt \int_{x-d_1(x)}^{x}\frac{dt}{r(t)} < 2r(x)v'(x).
\end{aligned}$$

Thus we have proved (2.7). Further, as in [3], by (2.6), (2.2) and (2.7) we obtain (2.8):

$$\frac{\rho(x)}{h(x)} = \frac{\dfrac{1}{\varphi(x)} + \dfrac{1}{\psi(x)}}{\dfrac{r(x)v'(x)}{v(x)} + \dfrac{r(x)|u'(x)|}{u(x)}} \le \max\left\{\frac{v(x)}{r(x)v'(x)}\ \frac{1}{\varphi(x)}, \frac{u(x)}{r(x)|u'(x)|}\ \frac{1}{\psi(x)}\right\} \le 2,$$

$$\frac{\rho(x)}{h(x)} = \frac{\dfrac{1}{\varphi(x)} + \dfrac{1}{\psi(x)}}{\dfrac{r(x)v'(x)}{v(x)} + \dfrac{r(x)|u'(x)|}{u(x)}} \ge \min\left\{\frac{v(x)}{r(x)v'(x)}\ \frac{1}{\varphi(x)}, \frac{u(x)}{r(x)|u'(x)|}\ \frac{1}{\psi(x)}\right\} \ge \frac{1}{2}.$$

From (2.8) and (1.8) one deduces (2.9). □

Proof of Corollary 2.2.1. From (1.2), (1.3) and (2.10) it follows that for $d \gg 1$, one has

$$ad\int_{x-d}^{x} q(\xi)d\xi > 1,\ ad\int_{x}^{x+d} q(\xi)d\xi > 1,\ ad\int_{x-d}^{x+d} q(\xi)d\xi > 2, \quad x \in R. \tag{4.6}$$

For a fixed $x \in R$ consider the function $S(d)$, $d \ge 0$ (see (2.11)). From (2.11) and (4.6) it follows that $S(0) = 0$, $S(\infty) = \infty$. Since $S(d)$ does not decrease on R_+, there exists a unique solution $\tilde{d}(x)$ to the equation $S(d) = 2$. Further, one has an analogue of (4.3) for $u(x)$ for $s \ge 0$:

$$\begin{aligned}
r(x)|u'(x)|\int_x^{x+s}\frac{dt}{r(t)} &= u(x) - u(x+s) \\
&\quad + \int_x^{x+s}\frac{1}{r(t)}\int_x^{t} q(\xi)u(\xi)d\xi dt, \quad x \in R.
\end{aligned} \tag{4.7}$$

In (4.3), (4.7) set $s = \tilde{d}(x)$. Then by (2.2), (2.10), and the definition of $\tilde{d}(x)$, one has

$$\begin{aligned}
\frac{1}{\rho(x)} &= \frac{r(x)v'(x)}{v(x)} + \frac{r(x)|u'(x)|}{u(x)} < \left(\int_{x-\tilde{d}(x)}^{x}\frac{dt}{r(t)}\right)^{-1} + \left(\int_x^{x+\tilde{d}(x)}\frac{dt}{r(t)}\right)^{-1} \\
&\quad + \int_{x-\tilde{d}(x)}^{x+\tilde{d}(x)} q(t)dt \le \frac{2+2a}{\tilde{d}(x)}.
\end{aligned}$$

Let us proceed as in the proof of (4.5), but integrate by $s \in [0, \tilde{d}(x)]$. We obtain

$$v(x)\int_{x-\tilde{d}(x)}^{x} q(t)dt < r(x)v'(x)\left[1+\int_{x-\tilde{d}(x)}^{x}\frac{dt}{r(t)}\cdot\int_{x-\tilde{d}(x)}^{x} q(t)dt\right]$$
$$\le r(x)v'(x)\left[1+a\tilde{d}(x)\int_{x-\tilde{d}(x)}^{x} q(t)dt\right],$$

$$u(x)\int_{x}^{x+\tilde{d}(x)} q(t)dt < r(x)|u'(x)|\left[1+\int_{x}^{x+\tilde{d}(x)} q(t)dt\cdot\int_{x}^{x+\tilde{d}(x)}\frac{dt}{r(t)}\right]$$
$$\le r(x)|u'(x)|\left[1+a\tilde{d}(x)\int_{x}^{x+\tilde{d}(x)} q(t)dt\right].$$

These inequalities, the definition of $\tilde{d}(x)$ and (2.2) imply

$$\frac{2}{\tilde{d}(x)} = \int_{x-\tilde{d}(x)}^{x+\tilde{d}(x)} q(t)dt$$
$$< \left[\frac{r(x)v'(x)}{v(x)}+\frac{r(x)|u'(x)|}{u(x)}\right]\left[1+a\tilde{d}(x)\int_{x-\tilde{d}(x)}^{x+\tilde{d}(x)} q(t)dt\right] = \frac{2a+1}{\rho(x)}.$$

□

5. Inversion of a non-homogeneous Sturm-Liouville equation in $L_p(R)$

In this section we prove Theorem 2.3.

Proof of Lemma 2.3. Clearly, $2\Phi_1(d) \ge \Phi_2(d)$ for $d \ge 0$, where

$$\Phi_1(d) = \int_{x-d}^{x+d}\frac{dt}{r(t)h(t)}\ , \quad \Phi_2(d) = \int_{x-d}^{x+d}\frac{dt}{r(t)\rho(t)}\ , \qquad x \in R$$

(see (2.8)). Since by (2.4) one has $\Phi_2(\infty) = \infty$, we obtain $\Phi_1(0) = 0,\ \Phi_1(\infty) = \infty$ and $\Phi_1(d)$ is monotone increasing on $(0, \infty)$. Therefore, the equation $\Phi_1(d) = 1$ has a unique, finite positive solution. Let $d(x)$ be this solution. Then,

$$|d(x+s) - d(x)| \le |s| \qquad \text{for} \qquad |s| \le d(x),\ x \in R. \tag{5.1}$$

Indeed, let $s \in [0, d(x)]$. Clearly,

$$1 \le \int_{(x+s)-(d(x)+s)}^{(x+s)+(s+d(x))}\frac{d\xi}{r(\xi)h(\xi)}\ , \qquad 1 \ge \int_{(x+s)-(d(x)-s)}^{(x+s)+(d(x)-s)}\frac{d\xi}{r(\xi)h(\xi)}.$$

By the definition of $d(x)$, taking into account that $\Phi_1(d)$ is monotone, these inequalities imply respectively $d(x+s) \le d(x) + s,\ d(x+s) \ge d(x) - s$, i.e., (5.1) holds. The case $s \in [-d(x), 0]$ can be treated similarly. By (5.1), $d(x)$ is continuous. □

Lemma 5.1. *Let $x \in R,\ t \in [x - d(x), x + d(x)]$. Then*

$$\alpha^{-1}v(x) \le v(t) \le \alpha v(x),\quad \alpha^{-1}u(x) \le u(t) \le \alpha u(x),\quad \alpha = \exp(2), \tag{5.2}$$

$$\alpha^{-1}\rho(x) \le \rho(t) \le \alpha\rho(x),\quad (4\alpha)^{-1}h(x) \le h(t) \le (4\alpha)h(x). \tag{5.3}$$

Proof. For $t \in [x, x+d(x)]$ from (2.7) and (2.6), it follows that

$$\ell n \frac{v(t)}{v(x)} \le 2\int_x^t \frac{d\xi}{r(\xi)\varphi(\xi)} < 2\int_x^t \frac{d\xi}{r(\xi)h(\xi)} < 2\int_{x-d(x)}^{x+d(x)} \frac{d\xi}{r(\xi)h(\xi)} = 2.$$

For $t \in [x-d(x), x]$ we similarly get

$$\ell n \frac{v(x)}{v(t)} \le 2\int_t^x \frac{d\xi}{r(\xi)\varphi(\xi)} \le 2\int_t^x \frac{d\xi}{r(\xi)h(\xi)} < 2\int_{x-d(x)}^{x+d(x)} \frac{d\xi}{r(\xi)h(\xi)} = 2.$$

These inequalities and (2.2) imply (5.2) for $v(x)$. For $u(x)$ one can check (5.2) in a similar way. Further, from (2.2) and (5.2) we obtain

$$\begin{aligned} &\frac{p(t)}{p(x)} = \frac{u(t)}{u(x)}\cdot\frac{v(t)}{v(x)} \le \frac{v(t)}{v(x)} \le \alpha \qquad \text{for} \quad t \in [x, x+d(x)], \\ &\frac{p(t)}{p(x)} = \frac{u(t)}{u(x)}\cdot\frac{v(t)}{v(x)} \ge \frac{u(t)}{u(x)} \ge \alpha^{-1} \qquad \text{for} \quad t \in [x, x+d(x)]. \end{aligned} \tag{5.4}$$

For $t \in [x-d(x), x]$ estimates (5.3) for $\rho(\cdot)$ can be obtained similarly. Estimates (5.3) for $h(x)$ follow from estimates (5.4) for $\rho(x)$ and (2.8). □

Definition 5.1. We say that a system of segments $\{\Delta_n\}_{n\in N'}$, $N' = \{\pm 1, \pm 2, \dots\}$ forms an $R(x)$-covering of R if the following assertions hold:

(1) $\Delta_n = [\Delta_n^-, \Delta_n^+] \overset{\text{def}}{=} [x_n - d(x_n), x_n + d(x_n)], \quad n \in N'$.
(2) $\Delta_{n+1}^- = \Delta_n^+$ if $n \ge 1$; $\Delta_{n-1}^+ = \Delta_n^-$ if $n \le -1$.
(3) $\Delta_1^- = \Delta_{-1}^+ = x$; $\bigcup_{n\ne 0} \Delta_n = R$.

Lemma 5.2. *For every $x \in R$ there exists an $R(x)$-covering of R.*

Proof. Let $\mu(t) = t - d(t) - x$. Then $\mu(x) = -d(x) < 0$. Let us verify the $\mu(t_0) > 0$ for some $t_0 > x$. Assume the contrary. Then $\mu(t) = t - d(t) - x \le 0$ for all $t > x$. From this, by the definition of $d(t)$, (2.8) and (2.4), we get a contradiction:

$$1 = \int_{t-d(t)}^{t+d(t)} \frac{d\xi}{r(\xi)h(\xi)} > \int_x^t \frac{d\xi}{r(\xi)h(\xi)} \ge \frac{1}{2}\int_x^t \frac{d\xi}{r(\xi)\rho(\xi)} \to \infty \ \text{ as } \ t \to \infty.$$

Hence $\mu(t_1) = 0$ for some $t_1 > x$, and we set $\Delta_1 = [t_1 - d(t_1), t_1 + d(t_1)]$. Similarly, we construct $\Delta_n, n \ge 2$. Let us verify that $\bigcup_{n\ge 1}^{\infty} \Delta_n = [x, \infty)$. If this is not the case, then there exists z such that $x_n + d(x_n) < z$ for all $n \ge 1$. The sequence $\{x_n\}_{n=1}^{\infty}$ is monotone increasing and bounded and, therefore, it converges to some x_0. Obviously, $\infty > z - x > 2\sum_{n=1}^{\infty} d(x_n)$, therefore, $d(x_n) \to 0$ as $n \to \infty$. Since $d(x)$ is continuous, one has $d(x_0) = 0$, a contradiction to Lemma 2.3. Similarly, one can construct the covering to the left from x. □

Remark. Assertions similar to Lemma 5.2 were used by M. Otelbaev (see [7]).

Lemma 5.3. *Suppose that* (1.2) *and* (2.1) *hold. Then,*

$$\sup_{x\in R} T(x) \le 1, \quad T(x) \overset{\text{def}}{=} \int_{-\infty}^{\infty} q(t)G(x,t)\, dt, \quad x \in R. \tag{5.5}$$

Proof. From (2.2) it follows that for $x \in R$:

$$r(x)v'(x) \ge \int_{-\infty}^{x} q(\xi)v(\xi)d\xi, \quad r(x)|u'(x)| \ge \int_x^{\infty} q(\xi)u(\xi)d\xi. \tag{5.6}$$

Now using (5.6) and once more (2.2) we get

$$\begin{aligned} 1 &= r(x)[v'(x)u(x) - u'(x)v(x)] \\ &\geq u(x)\int_{-\infty}^{x} q(\xi)v(\xi)d\xi + v(x)\int_{x}^{\infty} q(\xi)u(\xi)d\xi = T(x). \end{aligned}$$

$\square$

Lemma 5.4. *Let $\mathcal{A} > 0$ (see (1.10)). Then*

$$H \overset{\text{def}}{=} \sup_{x\in R}\int_{-\infty}^{\infty} G(x,t)\ dt \leq \tau\mathcal{A}^{-1}. \tag{5.7}$$

Proof. Below we use Lemmas 5.2 (and its notation), 5.1 and 5.3:

$$\begin{aligned} 1 \geq T(x) &= u(x)\sum_{n=-\infty}^{-1}\int_{\Delta_n} q(t)v(t)dt + v(x)\sum_{n=1}^{\infty}\int_{\Delta_n} q(t)u(t)dt \\ &\geq \frac{2}{\alpha}\ u(x)\sum_{n=-\infty}^{-1} v(x_n)d(x_n)\left[\frac{1}{2d(x_n)}\int_{\Delta_n} q(t)dt\right] \\ &+ \frac{2}{\alpha}\ v(x)\sum_{n=-\infty}^{-1} u(x_n)d(x_n)\left[\frac{1}{2d(x_n)}\int_{\Delta_n} q(t)dt\right] \\ &\geq \frac{\mathcal{A}}{\alpha}\left\{u(x)\sum_{n=-\infty}^{-1}\int_{\Delta_n} v(t)\frac{v(x_n)}{v(t)}\ dt + v(x)\sum_{n=1}^{\infty}\int_{\Delta_n} u(t)\cdot\frac{u(x_n)}{u(t)}\ dt\right\} \\ &\geq \frac{\mathcal{A}}{\alpha^2}\left\{u(x)\sum_{n=-\infty}^{-1}\int_{\Delta_n} v(t)dt + v(x)\sum_{n=1}^{\infty}\int_{\Delta_n} u(t)dt\right\} = \frac{\mathcal{A}}{\alpha^2}\int_{-\infty}^{\infty} G(x,t)\ dt. \end{aligned}$$

$\square$

Lemma 5.5. *Let $p \in [1,\infty]$. Consider $G\colon L_p(R) \to L_p(R)$ (see (1.4)). If $H < \infty$ (see (5.7)), then $\|G\|_{p\to p} \leq H$. In particular, if $\mathcal{A} > 0$ (see (1.10), then $\|G\|_{p\to p} \leq \tau\mathcal{A}^{-1}$.*

Proof. The operator $G\colon L_1(R) \to L_1(R)$ is bounded. Indeed, since $G(x,t) = G(t,x)$ for $x,t \in R$, by Fubini's theorem, using $H < \infty$, we get

$$\begin{aligned} \|(Gf)\|_1 &\leq \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} G(x,t)|f(t)|dt\right)dx \\ &= \int_{-\infty}^{\infty} |f(t)|\left(\int_{-\infty}^{\infty} G(x,t)dx\right)dt \leq H\|f\|_1. \end{aligned}$$

Hence, since $G(x,t) = G(t,x)$, the operator $G\colon L_\infty(R) \to L_\infty(R)$ is also bounded, $\|G\|_{\infty\to\infty} \leq H$. It remains to apply the Riesz-Torin theorem [14, Ch. XII, §1]. $\square$

Lemma 5.6. *One has $G(x,t) \leq 2h(x),\ x,t \in R$.*

Proof. By (2.2) and (2.8) we get

$$G(x,t) = \rho(x)\begin{cases} v(t)v(x)^{-1}, & x \geq t, \\ \\ u(t)u(x)^{-1}, & x \geq t, \end{cases} \leq \rho(x) \leq 2h(x). \qquad \square$$

Lemma 5.7. *Let $p \in [1,\infty]$, $H < \infty$ (see (5.7)) and suppose that (1.7) has no solutions $z(x) \in L_p(R)$ apart from $z(x) \equiv 0$. Then assertions 1)-2) from §1 hold.*

Proof. Let $y = (Gf)(x)$. Below we apply Hölder's inequality and Lemma 5.6:

$$|y(x)| < \left(\int_{-\infty}^{\infty} G(x,t)dt\right)^{1/p'} \left(\int_{-\infty}^{\infty} G(x,t)|f(t)|^p dt\right)^{1/p} \le \tau H^{1/p'} h(x)^{1/p} \|f\|_p.$$

Hence, for $x \in R$, the following integrals converge:

$$\int_{-\infty}^{x} v(t)f(t)dt\ , \qquad \int_{x}^{\infty} u(t)f(t)dt \tag{5.8}$$

and $y(x)$ is absolutely continuous for $x \in R$. Since

$$r(x)y'(x) = (r(x)u'(x)) \int_{-\infty}^{x} v(t)f(t)dt + (r(x)v'(x)) \int_{x}^{\infty} u(t)f(t)dt,$$

the functions $(r(x)u'(x)), (r(x)v'(x))$ are absolutely continuous and integrals (5.8) converge, we conclude that $r(x)y'(x)$ is absolutely continuous and, by (2.2), one has (1.1) almost everywhere. By Lemma 5.5, $\|G\|_{p\to p} \le H$ and, therefore, $y(x) \in L_p(R)$, and (1.1) has no other solutions in $L_p(R)$. □

Proof of Theorem 2.3. Below we apply Hölder's inequality, (5.5) and (5.7):

$$\begin{aligned}
\|q(x)^{1/p} y(x)\|_p^p &= \int_{-\infty}^{\infty} q(x) \left| \int_{-\infty}^{\infty} G(x,t)|f(t)|dt \right|^p dx \\
&\le \int_{-\infty}^{\infty} q(x) \left[\int_{-\infty}^{\infty} G(x,t)dt\right]^{p/p'} \left[\int_{-\infty}^{\infty} G(x,t)|f(t)|^p dt\right] dx \\
&\le (\tau\mathcal{A})^{-p/p'} \int_{-\infty}^{\infty} |f(t)|^p \left[\int_{-\infty}^{\infty} q(x)G(x,t)dx\right] dt \le (\tau\mathcal{A})^{-p/p'} \|f(x)\|_p^p.
\end{aligned}$$

□

Proof of Corollary 2.3.1. The proof follows from Theorem 2.3.

References

1. N. Chernyavskaya and L. Shuster, *On the WKB-method*, Differentsial'nye Uravneniya **25** (1989), 1826-1829. MR **90k:**34065
2. ———, *On a representation of the solution of the Sturm-Liouville equation and its applications*, Differentsial'nye Uravnenia **28** (1992), 537-540. MR **93f:**34088
3. ———, *Estimates for Green's function of the Sturm-Liouville operator*, J. Diff. Eq. **111** (1994), 410-420. MR **95i:**34044
4. ———, *Asymptotic integration of a nonoscillatory second order differential equation with linear perturbation*, preprint, AMS PPS # 199508-34-001 (1995).
5. ———, *Estimates for the Green function of a general Sturm-Liouville operator and their application*, preprint, AMS PPS # 1130-34-002 (1997).
6. E.B. Davies and E.M. Harrell, *Conformally flat Riemannian metrics, Schrödinger operators and semiclassical approximation*, J. Diff. Eq. **66** (1987), 165-188. MR **88a:**35061
7. K. Mynbaev and M. Otelbaev, *"Weighted Functional Spaces and Differential Operator Spectrum"*, Nauka, Moscow, 1988. MR **89h:**46036
8. R. Oinarov, *Properties of a Sturm-Liouville operator in L_p*, Izvestija Akad. Nauk Kazakh. SSR **1** (1990), 43-47. MR **91k:**47116
9. M. Sapenov and L. Shuster, *Estimates of Green's function and theorem on separability of a Sturm-Liouville operator in L_p*, Manuscript deposited at VINITI, No. 8257-B85.

10. L. Shuster, *A priori properties of solutions of a Sturm-Liouville equation and A.M. Molchanov's criterion*, Math. Notes Ac. Sci. USSR **50** (1991), 746-751, Translation: Matematicheskie Zametki. MR **92m:**34066
11. W.A. Steklov, *Sur une méthode nouvelle pour résoudre plusiers problems sur le développement d'une fonction arbitraire en séries infinies*, Compes Rendus, Paris **144** (1907), 1329-1332.
12. E.C. Titchmarsh, *The Theory of Functions*, Oxford, 1932.
13. E.T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge, 1958. MR **31:**2375
14. A. Zygmund, *Trigonometric Series*, Cambridge, 1959. MR **21:**6498

Department of Mathematics and Computer Science, Ben-Gurion University of the Negev, P.O.B. 653, Beer-Sheva, 84105, Israel

Department of Mathematics and Computer Science, Bar-Ilan University, Ramat-Gan 52900, Israel