

ON THE EXCESS OF SETS OF COMPLEX EXPONENTIALS

NOBUHIKO FUJII, AKIHIRO NAKAMURA, AND RAY REDHEFFER

(Communicated by J. Marshall Ash)

ABSTRACT. For $-\infty < n < \infty$ let μ_n be complex numbers such that $\mu_n - n$ is bounded. For $n > 0$ define $\lambda_n = \mu_n + a$, $\lambda_{-n} = \mu_{-n} - b$ where $a, b \geq 0$. Then the excesses E in the sense of Paley and Wiener satisfy $E(\{\lambda_n\}) \leq E(\{\mu_n\})$.

The original sources underlying the problem treated here are [1], [2], [4], and all the results used can be found in the excellent expository account [5], supplemented in a few cases by [3]. In view of this, we do not interrupt the flow by constant reference to the literature.

We use a, b, c for constants and we denote by $\mu = \{\mu_n\}$ a complex sequence defined for $-\infty < n < \infty$. The L^2 excess of the set μ in the sense of Paley and Wiener is denoted by $E(\mu)$. If $|\mu_n - n| \leq c$, it is well known that the completeness interval is $[-\pi, \pi]$ and that $E(\mu)$ on this interval is finite. Our object is to prove:

Theorem 1. *Let $|\mu_n - n| \leq c$ for $-\infty < n < \infty$ and $\lambda_0 = \mu_0$,*

$$\lambda_n = \mu_n + a, \quad \lambda_{-n} = \mu_{-n} - b, \quad n > 0,$$

where $a \geq 0$ and $b \geq 0$. Then $E(\lambda) \leq E(\mu)$ on the interval $[-\pi, \pi]$.

For example, suppose $E(\mu) = 0$, so μ is exact on $[-\pi, \pi]$; that is, $\{\exp(i\mu_n x)\}$ is complete as it stands but becomes incomplete if any term is removed. Then Theorem 1 shows that $\{\exp(i\lambda_n x)\}$ is either exact or incomplete. Hence it is *free* in the sense of Schwartz; that is, no exponential $\exp(i\lambda_n x)$ is in the subspace spanned by the others. This always holds if $c < 1/4$, since the set $\exp(i\mu_n x)$ forms a basis in that case.

We precede the proof by the following simplifications:

- (i) It is easily checked that the completeness properties are unchanged when λ_n is replaced by $\lambda_n - d$, where d is any real constant. Hence there is no loss of generality in taking $b = 0$ and replacing a by $a_0 = a + b$. We write a_0 instead of $a + b$ because we want a to be a variable, $0 \leq a \leq a_0$. This interval describes the *relevant values* of a .
- (ii) Since any finite number of terms can be changed without altering the completeness, we assume without loss of generality that

$$\operatorname{Re} \mu_n < 0 \text{ for } n < 0, \quad \operatorname{Re} \mu_n \geq 1 \text{ for } n > 0.$$

In particular, $\lambda_n \neq 0$ for $n \neq 0$ and all relevant values of a . This step and (iii) below may involve increasing c , but that does no harm.

Received by the editors January 31, 1997 and, in revised form, September 20, 1997.
1991 *Mathematics Subject Classification*. Primary 30B60.

- (iii) By adding or removing a finite number of terms μ_n and the λ_n corresponding thereto, we can ensure that the set μ is exact.

1. PROOF

Let

$$(1) \quad F(a, x) = \prod_{n=1}^{\infty} \left(1 - \frac{x}{\mu_{-n}}\right) \left(1 - \frac{x}{\mu_n + a}\right),$$

where the factors have been grouped in such a way that the product converges. Since μ is exact we have $F(0, x) \in L^2$, but this would not hold if we had retained the factor corresponding to μ_0 . Theorem 1 will follow as soon as we show $F(a, x) \in L^2$.

Let us therefore consider the integral

$$(2) \quad I(a) = \int_{-R}^{R+a} |F(a, x)|^2 dx,$$

where R is a large constant that will later tend to ∞ . The integral is decomposed in the form

$$(3) \quad I(a) = \int_{-R}^{-m} + \int_{-m}^{m+a} + \int_{m+a}^{R+a},$$

where $m \geq 2c + c^2$ is an integer. With $\mu_{-n} = -p - iq$ and $\mu_n = r + is$ for $n \geq m$, the constraint is

$$(4) \quad (p - n)^2 + q^2 \leq c^2, \quad (r - n)^2 + s^2 \leq c^2.$$

The magnitude squared of the general term in the product is

$$K(a) = \left|1 - \frac{x}{\mu_{-n}}\right|^2 \frac{(r + a - x)^2 + s^2}{(r + a)^2 + s^2}.$$

In the first integral (3) we have $x \leq -m$. Differentiation shows that $K'(a)$ has the same sign as $s^2 - (r + a)^2 + x(r + a)$, which is at most $s^2 - m$ and hence ≤ 0 . Therefore the first integral is dominated by the corresponding integral with μ , which converges when $R \rightarrow \infty$.

In the second integral $-m \leq x \leq m + a$. Since the integrand is a continuous function of (a, x) the integral is bounded by a constant C_m for all relevant values of a .

After the substitution $x = y + a$ the third integral is

$$J(a) = \int_m^R \prod_{n=1}^{\infty} \left|1 - \frac{y + a}{\mu_{-n}}\right|^2 \left|1 - \frac{y + a}{\mu_n + a}\right|^2 dy.$$

The monotonicity of the integrand

$$\left|\frac{\mu_n - y}{\mu_{-n}}\right|^2 \left|\frac{\mu_{-n} - a - y}{\mu_n + a}\right|^2$$

as a function of a is determined by the second factor,

$$L(a) = \left|\frac{p + iq + a + y}{r + is + a}\right|^2 = \frac{(p + a + y)^2 + q^2}{(r + a)^2 + s^2}.$$

Evidently $L'(a)$ has the same sign as

$$[(r + a)^2 + s^2](p + a + y) - [(p + a + y)^2 + q^2](r + a).$$

Now comes an important point. Other variables being fixed, $L'(a)$ is largest when $q = 0$. Since $q = 0$ allows the greatest freedom in the choice of p , we see that $q = 0$ is in fact the worst case. Setting $q = 0$, we divide out the factor $p + a + y$ and find that $L'(a) \leq 0$ holds if the function

$$P = (r + a)^2 + s^2 - (p + a + y)(r + a) = (r + a)(r - p - y) + s^2$$

satisfies $P \leq 0$. The two points (p, q) and (r, s) are both in a circle of radius c centered at $(n, 0)$. Hence their distance is at most $2c$, so $|r - p| \leq 2c$. Since also $|s| \leq c$, $r \geq 1$ and $y \geq m \geq 2c + c^2$, it follows that $P \leq 0$. Hence $L'(a) \leq 0$ and the integral is maximized when $a = 0$. In that case it reduces to the corresponding integral with μ , which converges as $R \rightarrow \infty$. This completes the proof.

2. SUPPLEMENTARY REMARKS

The foregoing methods apply to the weighted integral

$$\int_{-\infty}^{\infty} |F(a, x)|^2 w(x) dx,$$

where $w(x)$ is any positive continuous function satisfying

$$0 < \liminf_{|x| \rightarrow \infty} \frac{w(x + d)}{w(x)} \leq \limsup_{|x| \rightarrow \infty} \frac{w(x + d)}{w(x)} < \infty, \quad -\infty < d < \infty.$$

Namely, if the integral converges for $a = 0$ then it converges for $a \geq 0$. Admissible weights that are often encountered are

$$w(x) = e^{hx}, \quad w(x) = (1 + x^2)^h,$$

where h is a positive or negative constant.

As another generalization, instead of $|\mu_n - n| \leq c$ we could assume that $E(\mu)$ is finite and that

$$n\text{Re } \mu_n \geq 0, \quad |\text{Im } \mu_n| \leq c, \quad |\mu_{-n} + \mu_n| \leq c.$$

The question whether $|E(\mu)| < \infty \Rightarrow E(\lambda) \leq E(\mu)$ was suggested to one of us by Prof. Lennart Carleson. It appears to be considerably deeper than Theorem 1 and is left open here. The case $p \neq 2$ is also left open, though we do have

$$E(\lambda) \leq E(\mu) + 1$$

for $1 \leq p \leq \infty$. This follows from the result for $p = 2$ and from the fact that changing p changes $E(\lambda)$ by at most 1.

Theorem 1 is especially plausible when when $\mu = \{n\}$ and $b = 0$. In that case the set λ for $a = 1$ is the same as $\mu - \{1\}$, the set for $a = 2$ is the same as $\mu - \{1, 2\}$ and so on. Further insight into the case $\mu = \{n\}$ is given by [1], Theorem V and its proof. The latter suggests a conjecture that is formulated next.

An exponent p satisfying $1 < p < \infty$ will be called *critical* if

$$(5) \quad 1 < r < p < q < \infty \Rightarrow E(\lambda, q) > E(\lambda, r).$$

When $p = 1$ or $p = \infty$ we modify (5) somewhat and instead of L^∞ we use the class of absolutely continuous functions that vanish at $\pm\pi$. As $a + b$ increases in Theorem 1, we conjecture that the critical value p in the excess $E(\lambda, p)$ increases steadily to ∞ , passing through all intermediate values on the way. Upon further increase $E(\lambda, p)$ drops by 1 while p changes to 1, and the process repeats.

REFERENCES

- [1] Levinson, Norman, Gap and Density Theorems, AMS Colloquium Publication XXVI (1940), Chapters I, III and IV. MR **2**:180d
- [2] Paley, Raymond E. A. C and Norbert Wiener, Fourier Transforms in the Complex Domain, AMS Colloquium Publication XIX (1934), Chapter VI. CMP 97:13
- [3] Redheffer, Raymond M., Completeness of Sets of Complex Exponentials, Advances in Mathematics **24** (1977) 1–62. MR **56**:5852
- [4] Schwartz, Laurent, Approximation d'une fonction quelconque par des sommes d'exponentielles imaginaires, Ann. Fac. Sci. Toulouse (1943), 111–176. Reprint (Paris 1959) with some additions.
- [5] Young, Robert M., An Introduction to Nonharmonic Fourier Series, Academic Press 1980, Chapter 3. MR **81m**:42027

DEPARTMENT OF MATHEMATICS, TOKAI UNIVERSITY, 3-20-1 ORIDO, SHIMIZU, SHIZUOKA 424-8610, JAPAN

E-mail address: `nfujii@scc.u-tokai.ac.jp`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA, 90095-1555