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HOMOGENEITY AND THE DISJOINT ARCS PROPERTY

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(Communicated by Alan Dow)

Dedicated to my wife Ewa

Abstract. Some previous results of the author towards a classification of homogeneous metric continua are improved. The disjoint arcs property is fully revealed in this context. In particular, closed n-manifolds, $n = 1, 2$, are characterized as those homogeneous continua which do not have the disjoint arcs property.

1. Introduction

All spaces in this paper are metric separable and all mappings are continuous. A space X is said to be *homogeneous*, if for any two points $x, y \in X$ there exists a homeomorphism $h : X \to Y$ such that $h(x) = y$. A space (X, ρ) has the *disjoint arcs property (DAP)*, if for each $\epsilon > 0$ and for any two mappings $f, g : I = [0, 1] \to X$ there exist mappings $f', g' : I \to X$ satisfying $f'(I) \cap g'(I) = \emptyset$ and $\hat{\rho}(f, f') < \epsilon, \hat{\rho}(g, g') < \epsilon$, where $\hat{\rho}$ denotes the sup-metric induced by ρ.

Relationships between homogeneity and the disjoint arcs property were studied in a series of papers [3]–[5]. It was observed in [3] that all homogeneous locally compact, locally connected spaces of dimension greater than 2 have the DAP. In [4] special attention was paid to homogeneous curves which were proven to have the DAP, except for a simple closed curve. It turned out in [5] that all homogeneous locally connected continua, with the exception of n-manifolds, $n = 1, 2$, have the DAP. There exist interesting examples of 2-dimensional locally connected homogeneous continua different from surfaces. We know only one such continuum in R^3 described, e.g., in [2] or [6], and called the Pontryagin sphere (because its construction is similar to the original one of the Pontryagin disc in R^4).

In the present paper we prove, among other things, that, actually, all homogeneous continua with the exception of n-manifolds, $n = 1, 2$, have the DAP.

Let us recall four propositions which will be used or strengthened later in this paper.

Proposition 1.1 ([1, 4]). *Suppose X is a one-dimensional locally compact locally connected space with no local separating points. Then either X is a Menger curve*

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manifold or else X contains an open nonempty subset homeomorphic to an open subset of the Sierpiński universal planar curve.

In what follows a *solenoid* is an inverse limit of a sequence of simple closed curves with covering bonding maps.

Proposition 1.2 ([4])**.** *If X is a homogeneous continuum, then one of the following four cases holds.*

(i) *X is a solenoid.*
(ii) *X is locally connected and not a simple closed curve.*
(iii) *There exists an $\alpha > 0$ such that each nonempty open subset of X of diameter less than α consists of uncountably many nowhere dense components which are locally connected and have no local separating points.*
(iv) *Each component C of an arbitrary nonempty open subset of X has an open cover by subsets of C consisting of uncountably many components that are nowhere dense in C.*

If, additionally, X is 1-dimensional, then X is the Menger universal curve in case (ii), while in case (iii) the components are Menger curve manifolds.

Proposition 1.3 ([5])**.** *Let X be a locally connected continuum. Then the following statements are equivalent.*

(i) *X has no local separating points and no open nonempty subset of X is planar.*
(ii) *X has the DAP.*

Proposition 1.4 ([5])**.** *If X is a locally connected homogeneous continuum, then X is not an n-manifold, $n = 1, 2$, if and only if X has the DAP.*

2. Results

We need an extension of Proposition 1.3 to locally compact spaces.

Proposition 2.1. *Let X be a connected, locally connected, locally compact space. Then the following statements are equivalent.*

(i) *X has no local separating points and no nonempty open subset of X is planar.*
(ii) *X has the DAP.*

Proof. If X is compact, then the proposition is covered by Proposition 1.3. Assume X is not compact. Let $X' = X \cup \{\omega\}$ be a one-point compactification of X contained in the Hilbert cube Q. Let $f, g : I \to X$ be two mappings with intersecting images. Take an open connected neighborhood U of ω in Q such that its closure $\operatorname{cl} U$ is homeomorphic to Q and is disjoint with the continuum $f(I) \cup g(I)$. Observe that the space $Y = X' \cup \operatorname{cl} U$ is a locally connected continuum with no local separating points and no planar open nonempty subsets. It follows from Proposition 1.3 that Y has the DAP. If $f', g' : I \to Y$ are two mappings with disjoint images which are close enough to f and g, then

$$f'(I) \cup g'(I) \subset X.$$

This shows the DAP for X.

The proof of implication (ii) $\Rightarrow$ (i) is the same as the corresponding proof of Proposition 1.3 (see [5, pp. 84–85]). $\square$

Proposition 2.2. *Suppose X is a nondegenerate, connected, locally connected, locally compact space with no local separating points, containing no open subset homeomorphic to an open 2-disk. Then either*

(1) *X contains a nonempty open subset homeomorphic to an open subset of the Sierpiński universal planar curve or*
(2) *X has the DAP.*

Proof. Suppose case (1) does not hold. Observe that X does not contain a nonempty open planar subset. Indeed, if $U \subset X$ were such a set, then $\dim U = 1$, because there is no 2-disk in U, and, by Proposition 1.1, U would contain a nonempty open subset homeomorphic to an open subset of the Sierpiński curve.

By Proposition 2.1, X has the DAP. $\square$

Theorem 2.3. *If X is a homogeneous continuum, then one of the following six cases holds.*

(1) *X is a solenoid.*
(2) *X is a closed 2-manifold.*
(3) *X is locally connected and has the DAP.*
(4) *There exists an $\alpha > 0$ such that each nonempty open subset of X of diameter less than α consists of uncountably many nowhere dense components which are 2-manifolds.*
(5) *There exists an $\alpha > 0$ such that each nonempty open subset of X of diameter less than α consists of uncountably many nowhere dense components which are locally connected and have the DAP.*
(6) *Each component C of an arbitrary nonempty open subset of X has an open cover by subsets of C consisting of uncountably many components that are nowhere dense in C.*

Proof. In view of Proposition 1.2, it remains to split its case (ii) into (2) and (3), and (iii) into (4) and (5).

If (ii) is satisfied and X is not a 2-manifold, then it has the DAP by Proposition 1.4.

Assume case (iii) holds. Let C be a component of a nonempty open subset U of X of diameter less than α. If C contains an open 2-disk as its open subset, then, since X is homogeneous and components of U are locally connected, every component of U is a 2-manifold and (4) is satisfied. If C contains no open subset homeomorphic to an open 2-disk, then X has the DAP, because otherwise, by Proposition 2.2 and the homogeneity of X, C and all other components of U would be 1-dimensional, so X would be 1-dimensional. This is impossible, in view of Proposition 1.2. $\square$

The next result extends [4, Theorem 2.1] to arbitrary homogeneous continua different from solenoids and from 2-manifolds.

Theorem 2.4. *If X is a homogeneous continuum which is not a solenoid nor a 2-manifold, then either each component of an arbitrary open subset of X has the DAP or case (4) of Theorem 2.3 holds and X has the DAP.*

Proof. We consider all possible cases of Theorem 2.3.

First suppose case (3) holds. Then each component C of an arbitrary open subset of X is an open subset of X and the DAP for C easily follows from the DAP for X.

Suppose case (5) is satisfied. Let C be a component of an open subset U of X. If $f, g : I \to C$ are two mappings with intersecting images, then one can cover the locally connected continuum $A = f(I) \cup g(I)$ by open subsets $G_1, \ldots, G_n$ of U such that $\operatorname{diam} G_i < \alpha$ and $G_i \cap A$ is connected for $i = 1, \ldots, n$. Denote by D_i the component of G_i containing $G_i \cap A$. Observe that the subset $D = D_1 \cup \cdots \cup D_n$ of C is connected, locally connected and locally compact. In order to verify that D has the DAP, it is convenient to use Proposition 2.1, since each D_i, and thus D, has no local separating point and no open nonempty planar subsets. Hence C also has the DAP.

Case (6) can be treated in the same way as case (3) in the proof of [4, Theorem 2.1], and then we again get the DAP for any component of an arbitrary open subset U of X.

Finally, if (4) is satisfied, then X has the DAP by the same argument as in the preceding paragraph, restricted to $U = X$. □

The following interesting characterization is an immediate corollary to Theorem 2.4. and it extends [5, Theorem 2] to arbitrary continua.

Theorem 2.5. *Let $n = 1, 2$. A continuum X is a closed n-manifold if and only if X is homogeneous and does not have the DAP.*

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