

## HOMOGENEITY AND THE DISJOINT ARCS PROPERTY

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(Communicated by Alan Dow)

*Dedicated to my wife Ewa*

ABSTRACT. Some previous results of the author towards a classification of homogeneous metric continua are improved. The disjoint arcs property is fully revealed in this context. In particular, closed  $n$ -manifolds,  $n = 1, 2$ , are characterized as those homogeneous continua which do not have the disjoint arcs property.

### 1. INTRODUCTION

All spaces in this paper are metric separable and all mappings are continuous. A space  $X$  is said to be *homogeneous*, if for any two points  $x, y \in X$  there exists a homeomorphism  $h : X \rightarrow X$  such that  $h(x) = y$ . A space  $(X, \rho)$  has the *disjoint arcs property (DAP)*, if for each  $\epsilon > 0$  and for any two mappings  $f, g : I = [0, 1] \rightarrow X$  there exist mappings  $f', g' : I \rightarrow X$  satisfying  $f'(I) \cap g'(I) = \emptyset$  and  $\hat{\rho}(f, f') < \epsilon, \hat{\rho}(g, g') < \epsilon$ , where  $\hat{\rho}$  denotes the sup-metric induced by  $\rho$ .

Relationships between homogeneity and the disjoint arcs property were studied in a series of papers [3]–[5]. It was observed in [3] that all homogeneous locally compact, locally connected spaces of dimension greater than 2 have the DAP. In [4] special attention was paid to homogeneous curves which were proven to have the DAP, except for a simple closed curve. It turned out in [5] that all homogeneous locally connected continua, with the exception of  $n$ -manifolds,  $n = 1, 2$ , have the DAP. There exist interesting examples of 2-dimensional locally connected homogeneous continua different from surfaces. We know only one such continuum in  $R^3$  described, e.g., in [2] or [6], and called the Pontryagin sphere (because its construction is similar to the original one of the Pontryagin disc in  $R^4$ ).

In the present paper we prove, among other things, that, actually, all homogeneous continua with the exception of  $n$ -manifolds,  $n = 1, 2$ , have the DAP.

Let us recall four propositions which will be used or strengthened later in this paper.

**Proposition 1.1** ([1, 4]). *Suppose  $X$  is a one-dimensional locally compact locally connected space with no local separating points. Then either  $X$  is a Menger curve*

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manifold or else  $X$  contains an open nonempty subset homeomorphic to an open subset of the Sierpiński universal planar curve.

In what follows a *solenoid* is an inverse limit of a sequence of simple closed curves with covering bonding maps.

**Proposition 1.2** ([4]). *If  $X$  is a homogeneous continuum, then one of the following four cases holds.*

- (i)  $X$  is a solenoid.
- (ii)  $X$  is locally connected and not a simple closed curve.
- (iii) There exists an  $\alpha > 0$  such that each nonempty open subset of  $X$  of diameter less than  $\alpha$  consists of uncountably many nowhere dense components which are locally connected and have no local separating points.
- (iv) Each component  $C$  of an arbitrary nonempty open subset of  $X$  has an open cover by subsets of  $C$  consisting of uncountably many components that are nowhere dense in  $C$ .

If, additionally,  $X$  is 1-dimensional, then  $X$  is the Menger universal curve in case (ii), while in case (iii) the components are Menger curve manifolds.

**Proposition 1.3** ([5]). *Let  $X$  be a locally connected continuum. Then the following statements are equivalent.*

- (i)  $X$  has no local separating points and no open nonempty subset of  $X$  is planar.
- (ii)  $X$  has the DAP.

**Proposition 1.4** ([5]). *If  $X$  is a locally connected homogeneous continuum, then  $X$  is not an  $n$ -manifold,  $n = 1, 2$ , if and only if  $X$  has the DAP.*

## 2. RESULTS

We need an extension of Proposition 1.3 to locally compact spaces.

**Proposition 2.1.** *Let  $X$  be a connected, locally connected, locally compact space. Then the following statements are equivalent.*

- (i)  $X$  has no local separating points and no nonempty open subset of  $X$  is planar.
- (ii)  $X$  has the DAP.

*Proof.* If  $X$  is compact, then the proposition is covered by Proposition 1.3. Assume  $X$  is not compact. Let  $X' = X \cup \{\omega\}$  be a one-point compactification of  $X$  contained in the Hilbert cube  $Q$ . Let  $f, g : I \rightarrow X$  be two mappings with intersecting images. Take an open connected neighborhood  $U$  of  $\omega$  in  $Q$  such that its closure  $\text{cl}U$  is homeomorphic to  $Q$  and is disjoint with the continuum  $f(I) \cup g(I)$ . Observe that the space  $Y = X' \cup \text{cl}U$  is a locally connected continuum with no local separating points and no planar open nonempty subsets. It follows from Proposition 1.3 that  $Y$  has the DAP. If  $f', g' : I \rightarrow Y$  are two mappings with disjoint images which are close enough to  $f$  and  $g$ , then

$$f'(I) \cup g'(I) \subset X.$$

This shows the DAP for  $X$ .

The proof of implication (ii)  $\Rightarrow$  (i) is the same as the corresponding proof of Proposition 1.3 (see [5, pp. 84–85]).  $\square$

**Proposition 2.2.** *Suppose  $X$  is a nondegenerate, connected, locally connected, locally compact space with no local separating points, containing no open subset homeomorphic to an open 2-disk. Then either*

- (1)  $X$  contains a nonempty open subset homeomorphic to an open subset of the Sierpiński universal planar curve or
- (2)  $X$  has the DAP.

*Proof.* Suppose case (1) does not hold. Observe that  $X$  does not contain a nonempty open planar subset. Indeed, if  $U \subset X$  were such a set, then  $\dim U = 1$ , because there is no 2-disk in  $U$ , and, by Proposition 1.1,  $U$  would contain a nonempty open subset homeomorphic to an open subset of the Sierpiński curve.

By Proposition 2.1,  $X$  has the DAP.  $\square$

**Theorem 2.3.** *If  $X$  is a homogeneous continuum, then one of the following six cases holds.*

- (1)  $X$  is a solenoid.
- (2)  $X$  is a closed 2-manifold.
- (3)  $X$  is locally connected and has the DAP.
- (4) There exists an  $\alpha > 0$  such that each nonempty open subset of  $X$  of diameter less than  $\alpha$  consists of uncountably many nowhere dense components which are 2-manifolds.
- (5) There exists an  $\alpha > 0$  such that each nonempty open subset of  $X$  of diameter less than  $\alpha$  consists of uncountably many nowhere dense components which are locally connected and have the DAP.
- (6) Each component  $C$  of an arbitrary nonempty open subset of  $X$  has an open cover by subsets of  $C$  consisting of uncountably many components that are nowhere dense in  $C$ .

*Proof.* In view of Proposition 1.2, it remains to split its case (ii) into (2) and (3), and (iii) into (4) and (5).

If (ii) is satisfied and  $X$  is not a 2-manifold, then it has the DAP by Proposition 1.4.

Assume case (iii) holds. Let  $C$  be a component of a nonempty open subset  $U$  of  $X$  of diameter less than  $\alpha$ . If  $C$  contains an open 2-disk as its open subset, then, since  $X$  is homogeneous and components of  $U$  are locally connected, every component of  $U$  is a 2-manifold and (4) is satisfied. If  $C$  contains no open subset homeomorphic to an open 2-disk, then  $X$  has the DAP, because otherwise, by Proposition 2.2 and the homogeneity of  $X$ ,  $C$  and all other components of  $U$  would be 1-dimensional, so  $X$  would be 1-dimensional. This is impossible, in view of Proposition 1.2.  $\square$

The next result extends [4, Theorem 2.1] to arbitrary homogeneous continua different from solenoids and from 2-manifolds.

**Theorem 2.4.** *If  $X$  is a homogeneous continuum which is not a solenoid nor a 2-manifold, then either each component of an arbitrary open subset of  $X$  has the DAP or case (4) of Theorem 2.3 holds and  $X$  has the DAP.*

*Proof.* We consider all possible cases of Theorem 2.3.

First suppose case (3) holds. Then each component  $C$  of an arbitrary open subset of  $X$  is an open subset of  $X$  and the DAP for  $C$  easily follows from the DAP for  $X$ .

Suppose case (5) is satisfied. Let  $C$  be a component of an open subset  $U$  of  $X$ . If  $f, g : I \rightarrow C$  are two mappings with intersecting images, then one can cover the locally connected continuum  $A = f(I) \cup g(I)$  by open subsets  $G_1, \dots, G_n$  of  $U$  such that  $\text{diam } G_i < \alpha$  and  $G_i \cap A$  is connected for  $i = 1, \dots, n$ . Denote by  $D_i$  the component of  $G_i$  containing  $G_i \cap A$ . Observe that the subset  $D = D_1 \cup \dots \cup D_n$  of  $C$  is connected, locally connected and locally compact. In order to verify that  $D$  has the DAP, it is convenient to use Proposition 2.1, since each  $D_i$ , and thus  $D$ , has no local separating point and no open nonempty planar subsets. Hence  $C$  also has the DAP.

Case (6) can be treated in the same way as case (3) in the proof of [4, Theorem 2.1], and then we again get the DAP for any component of an arbitrary open subset  $U$  of  $X$ .

Finally, if (4) is satisfied, then  $X$  has the DAP by the same argument as in the preceding paragraph, restricted to  $U = X$ .  $\square$

The following interesting characterization is an immediate corollary to Theorem 2.4. and it extends [5, Theorem 2] to arbitrary continua.

**Theorem 2.5.** *Let  $n = 1, 2$ . A continuum  $X$  is a closed  $n$ -manifold if and only if  $X$  is homogeneous and does not have the DAP.*

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