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THE FURUTA INEQUALITY WITH NEGATIVE POWERS

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ABSTRACT. Let $A, B \in B(H)$ be bounded linear operators on a Hilbert space H satisfying $O \le B \le A$. Furuta showed the operator inequality $(A^r B^p A^r)^{\frac{1}{q}} \le A^{\frac{p+2r}{q}}$ as long as positive real numbers p, q, r satisfy $p + 2r \le (1 + 2r)q$ and $1 \le q$. In this paper, we show this inequality is valid if negative real numbers p, q, r satisfy a certain condition. Also, we investigate the optimality of that condition.

1. INTRODUCTION

Let A, B be bounded linear operators on a Hilbert space H. The following operator inequality is well-known as the Heinz inequality. (In the case where $p = \frac{1}{2}$, the inequality is called the Löwner inequality, and it is known that there are counter-examples in the case where $p > 1$. See [1] for $p = 2$ and [12, p. 465] for $p > 1$.)

Proposition 1 ([10],[11]). *$O \le B \le A$ implies $B^p \le A^p$ for all $p \in [0, 1]$.*

Concerning the Heinz inequality, Chan and Kwong [1] conjectured that $O \le B \le A$ will imply $B^2 \le \left(BA^2B\right)^{\frac{1}{2}}$ and Furuta gave an affirmative answer as follows.

Proposition 2 ([4]). *$O \le B \le A$ implies*

$$(A^r B^p A^r)^{\frac{1}{q}} \le A^{\frac{p+2r}{q}} \tag{1}$$

and

$$B^{\frac{p+2r}{q}} \le (B^r A^p B^r)^{\frac{1}{q}} \tag{2}$$

as long as positive real numbers p, q, r satisfy

$$p + 2r \le (1 + 2r)q \quad \text{and} \quad 1 \le q. \tag{3}$$

The range of p, q, r which satisfy the condition (3) is as in Figure 1 and, in [13], the author proved that the condition (3) is optimal for the validity of the Furuta inequality by using 2×2 matrices.

The Furuta inequality has many applications. In the operator mean theory, the Furuta inequality was generalized to the grand Furuta inequality [8] and it extends the Ando-Hiai log-majorization theory. Also, the Furuta inequality was used in the p-hyponormal operator theory ([2], [9]) and the relative entropy theory ([6], [7]).

Recently, it is known that the Furuta inequality is valid for some negative real numbers p, q, r by Yoshino [14] and Fujii, Furuta, Kamei [3]. And, in this paper,

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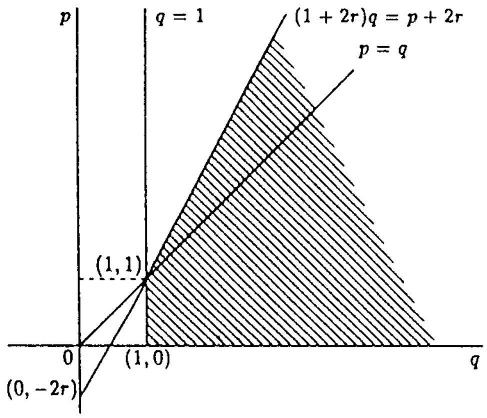

FIGURE 1

we shall consider the range of p, q, r for which the Furuta inequality is valid and investigate its optimality by using the same technique as in [13].

2. Results

Let $A, B \in B(H)$ be invertible operators on H satisfying $O \leq B \leq A$. In the Furuta inequality, if $p = 0$, then $A^0 = B^0 = I$ and the inequalities (1) and (2) become the equalities. Also, if $r = 0$, then the inequalities (1) and (2) reduce to the Löwner-Heinz inequality. Hence we may assume $p \neq 0$ and $r \neq 0$ in this paper. Since $O \leq B \leq A$ implies $O \leq A^{-1} \leq B^{-1}$, the validity of (1) is equivalent to the validity of (2). By a simple observation, the problem to find all real numbers p, q, r guaranteeing the validity of the Furuta inequality is reduced to the case where $0 < p, 0 < q$ and $r < 0$.

Theorem 3. *Let $A, B \in B(H)$ be invertible operators satisfying $O \leq B \leq A$. Let $0 < p \leq 1, 0 < q \leq 1$ and $-1 \leq 2r < 0$. Then*

$$(A^r B^p A^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}} \tag{1}$$

as long as real numbers p, q, r satisfy

$$-2r(1-q) \leq p \leq q - 2r(1-q) \tag{4}$$

and

$$\frac{-2r(1-q)-q}{1-2q} \leq p \leq \frac{-2r(1-q)}{1-2q}. \tag{5}$$

We need the following lemma to show Theorem 3.

Lemma 4 ([5]). *Let $A, B \in B(H)$ be positive operators and let λ be a real number. Then*

$$(ABA)^{\lambda} = AB^{\frac{1}{2}} \left(B^{\frac{1}{2}} A^2 B^{\frac{1}{2}}\right)^{\lambda-1} B^{\frac{1}{2}} A.$$

Proof of Theorem 3. Let $0 < p \leq 1, 0 < q \leq 1$ and $-1 \leq 2r < 0$.

First, assume $\frac{1}{2} \leq q \leq 1$ and p, q, r satisfy (4). (We remark that p, q, r satisfy (5) automatically in this case.) Then $O \leq A^{2r} \leq B^{2r}$ by the Heinz inequality. Since

$$0 \leq \frac{1}{q} - 1 \leq 1$$

and

$$0 \leq \frac{p+2r}{q} - 2r \leq 1,$$

we have

$$\begin{aligned}(A^r B^p A^r)^{\frac{1}{q}} &= A^r B^{\frac{p}{2}} \left(B^{\frac{p}{2}} A^{2r} B^{\frac{p}{2}}\right)^{\frac{1}{q}-1} B^{\frac{p}{2}} A^r \\ &\leq A^r B^{\frac{p}{2}} \left(B^{\frac{p}{2}} B^{2r} B^{\frac{p}{2}}\right)^{\frac{1}{q}-1} B^{\frac{p}{2}} A^r = A^r B^{\frac{p+2r}{q}-2r} A^r \\ &\leq A^r A^{\frac{p+2r}{q}-2r} A^r = A^{\frac{p+2r}{q}}\end{aligned}$$

by Lemma 4 and the Heinz inequality.

Second, assume $\frac{1}{3} \leq q < \frac{1}{2}$ and p, q, r satisfy (4) and (5). Let

$$p' = 2r, q' = \frac{q}{1-q}, r' = \frac{p}{2}$$

and let

$$p'' = -p', q'' = q', r'' = -r'.$$

Then

$$0 < p'' \leq 1, \frac{1}{2} \leq q'' < 1, -\frac{1}{2} \leq r'' < 0$$

and p'', q'', r'' satisfy

$$-2r''(1+q'') - q'' \leq p'' \leq -2r''(1+q'')$$

and

$$-2r''(1-q'') \leq p'' \leq q'' - 2r''(1-q'').$$

Hence, by the same arguments as in the case where $\frac{1}{2} \leq q \leq 1$, we have

$$\left(A^{r''} B^{p''} A^{r''}\right)^{\frac{1}{q''}} \leq A^{\frac{p''+2r''}{q''}},$$

and

(6) $$A^{\frac{p'+2r'}{q'}} \leq \left(A^{r'} B^{p'} A^{r'}\right)^{\frac{1}{q'}}.$$

It is known that (6) implies

$$\left(B^{r'} A^{p'} B^{r'}\right)^{\frac{1}{q'}} \leq B^{\frac{p'+2r'}{q'}}$$

and hence

$$\left(B^{\frac{p}{2}} A^{2r} B^{\frac{p}{2}}\right)^{\frac{1}{q}-1} \leq \left(B^{\frac{p}{2}} B^{2r} B^{\frac{p}{2}}\right)^{\frac{1}{q}-1}.$$

Since $0 \leq \frac{p+2r}{q} - 2r \leq 1$, we have

$$(A^r B^p A^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}}$$

by the same arguments as in the case where $\frac{1}{2} \leq q \leq 1$.

Similarly we can show the inequality (1) in the case where $\frac{1}{n+1} \leq q < \frac{1}{n}$ $(n = 3, 4, 5, \cdots)$ and p, q, r satisfy (4) and (5) because

$$\frac{-2r(1-q)-q}{1-2q} \leq -2r(1+q)-q < -2r(1+q) \leq \frac{-2r(1-q)}{1-2q}$$

if $0 < 2q \leq 1$ and $-1 \leq 2r < 0$. Thus the proof is complete.

Theorem 3 shows the validity of the Furuta inequality as long as p, q, r satisfy any one of the following conditions (I) , (II), (III), (IV).

Condition (I). $-\frac{1}{4} \leq r < 0$ and

$$\begin{cases} q \geq \frac{p+2r}{2r}, & \text{if} \quad 0 < p < -2r, \\ q > 0, & \text{if} \quad p = -2r, \\ q \geq \frac{p+2r}{2p+2r}, & \text{if} \quad -2r < p \leq \frac{1}{2}, \\ q \geq \frac{p+2r}{1+2r}, & \text{if} \quad \frac{1}{2} < p \leq 1. \end{cases}$$

Condition (II). $-\frac{1}{2} < r < -\frac{1}{4}$ and

$$\begin{cases} q \geq \frac{p+2r}{2r}, & \text{if} \quad 0 < p < \frac{1}{2}, \\ q \geq \frac{p+2r}{2p+2r-1}, & \text{if} \quad \frac{1}{2} \leq p < -2r, \\ q > 0, & \text{if} \quad p = -2r, \\ q \geq \frac{p+2r}{1+2r}, & \text{if} \quad -2r < p \leq 1. \end{cases}$$

Condition (III). $r = -\frac{1}{2}$ and

$$\begin{cases} q \geq \frac{p-1}{-1}, & \text{if} \quad 0 < p < \frac{1}{2}, \\ q \geq \frac{1}{2}, & \text{if} \quad \frac{1}{2} \leq p < 1, \\ q > 0, & \text{if} \quad p = 1. \end{cases}$$

Condition (IV). $r < -\frac{1}{2}$ and

$$q \geq 1, \qquad \text{if} \quad 0 < p \leq 1.$$

Remark. Yoshino [14] showed the validity of the inequality (1) as long as p, q, r satisfy $-1 < 2r < 0,\ 1 \leq 2q, (1+2r)q \leq p + 2r, 0 < p \leq 1$ or the condition (IV). Moreover, Fujii, Furuta, Kamei [3] showed the validity of the inequality (1) as long as p, q, r satisfy $-1 < 2r < 0,\ 1 \leq 2p \leq 2,\ 0 \leq p + 2r,\ (1+2r)q \leq p + 2r$.

In the sequel, we shall show that the condition (I) with $0 < p \leq -2r, \frac{1}{2} \leq p \leq 1$ and the conditions (II), (III), (IV) are optimal. Unfortunately, we remark that this is a partial answer for our problem and it is still an open question whether the Furuta inequality is valid as long as p, q, r satisfy

$$-\frac{1}{4} < r < 0, \quad -2r < p < \frac{1}{2}, \quad \frac{p+2r}{1+2r} \leq q < \frac{p+2r}{2p+2r}.$$

For our purpose, we need the following.

Lemma 5 ([13]). *Let* $a,\ b,\ d,\ \theta \in \mathbf{R}$ *satisfy* $0 < a + b,\ ab = d^2$ *where* $\mathbf{R}$ *denotes the set of all real numbers and let*

$$S = \begin{pmatrix} a & de^{-i\theta} \\ de^{i\theta} & b \end{pmatrix}.$$

Then

$$S^p = (a+b)^{p-1}S \quad \textit{for} \quad 0 < p.$$

Now we investigate the optimality of the conditions (I) with $0 < p \le -2r, \frac{1}{2} \le p \le 1$, (II), (III), (IV) by 6 steps.

Step 1. (I) $0 < p \le -2r$, (II) $0 < p < \frac{1}{2}$, (III) $0 < p < \frac{1}{2}$.

We show that if p, q, r satisfy

$$-\frac{1}{2} \le r < 0, 0 < p < -2r, 0 < q < \frac{p+2r}{2r},$$

then there exist invertible operators $A, B \in B(\mathbf{R}^2)$ with $O \le B \le A$ which do not satisfy the inequality (1).

Let

$$A = \begin{pmatrix} a & \sqrt{\varepsilon(a-b-\delta)} \\ \sqrt{\varepsilon(a-b-\delta)} & b+\varepsilon+\delta \end{pmatrix}$$

and let

$$B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix},$$

where

$$0 < b < 1 < a, 0 < \delta = \frac{1-b}{a-1}\varepsilon$$

as in [13]. Then $O \le B \le A$. To show the assertion by contradiction, we assume the inequality (1). We remark

$$a^{\frac{p+2r}{q}}, a^{\frac{2r}{q}} \to 0 \quad (a \to \infty).$$

Let

$$U = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} \sqrt{a-b-\delta} & \sqrt{\varepsilon} \\ \sqrt{\varepsilon} & -\sqrt{a-b-\delta} \end{pmatrix},$$

where

$$\gamma = a - b + \varepsilon - \delta.$$

Then U is unitary and

$$U^*AU = \begin{pmatrix} a+\varepsilon & 0 \\ 0 & b+\delta \end{pmatrix}$$

and hence, by (1) (see [13]), we have

$$\gamma^{-\frac{1}{q}} \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix}^{\frac{1}{q}} \le \begin{pmatrix} (a+\varepsilon)^{\frac{p+2r}{q}} & 0 \\ 0 & (b+\delta)^{\frac{p+2r}{q}} \end{pmatrix},$$

where

$$\begin{aligned} A_1 &= (a+\varepsilon)^{2r}(a-b-\delta+\varepsilon b^p), \\ A_2 &= (b+\delta)^{2r}(\varepsilon + b^p(a-b-\delta)), \\ A_3 &= (a+\varepsilon)^r(b+\delta)^r(1-b^p)\sqrt{\varepsilon(a-b-\delta)}. \end{aligned}$$

Let

$$V = \frac{1}{\sqrt{A_2 - A_1 + 2\varepsilon_1}} \begin{pmatrix} \sqrt{\varepsilon_1} & \sqrt{A_2 - A_1 + \varepsilon_1} \\ \sqrt{A_2 - A_1 + \varepsilon_1} & -\sqrt{\varepsilon_1} \end{pmatrix},$$

where

$$2\varepsilon_1 = -A_2 + A_1 + \sqrt{(A_2 - A_1)^2 + 4A_3^2}.$$

Then V is unitary and

$$V^* \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix} V = \begin{pmatrix} A_2 + \varepsilon_1 & 0 \\ 0 & A_1 - \varepsilon_1 \end{pmatrix}.$$

Hence

$$\begin{aligned} &\varepsilon_1 \left((A_2 + \varepsilon_1)^{\frac{1}{q}} - (a + \varepsilon)^{\frac{p+2r}{q}} \gamma^{\frac{1}{q}} \right) \left((b + \delta)^{\frac{p+2r}{q}} \gamma^{\frac{1}{q}} - (A_1 - \varepsilon_1)^{\frac{1}{q}} \right) \\ &\le (A_2 - A_1 + \varepsilon_1) \left((a + \varepsilon)^{\frac{p+2r}{q}} \gamma^{\frac{1}{q}} - (A_1 - \varepsilon_1)^{\frac{1}{q}} \right) \left((b + \delta)^{\frac{p+2r}{q}} \gamma^{\frac{1}{q}} - (A_2 + \varepsilon_1)^{\frac{1}{q}} \right) \end{aligned}$$

and

$$\begin{aligned} (7) \qquad & q \left(1 - a^{-1}\right) (1 - b^p)^2 \left(1 - a^{\frac{p+2r}{q}} b^{-\frac{p+2r}{q}}\right) \left(1 - a^{\frac{2r}{q}} b^{-\frac{p+2r}{q}}\right) \\ &\le a^{\frac{p+2r}{q} - 2r} b^{-\frac{p+2r}{q} + 2p + 2r - 1} \left(1 - a^{2r} b^{-p-2r}\right) \left(1 - a^{-\frac{p}{q}}\right) \\ &\times \left\{ p(1-b) \left(1 - a^{-1} b\right) \left(1 - a^{2r} b^{-p-2r}\right) \right. \\ &\left. \qquad - b^{1-p} \left(1 - b^p\right) \left(1 - a^{-1}\right) \left(1 - a^{2r} b^{-2r}\right) \right\} \end{aligned}$$

by similar arguments as in [13]. Since $\frac{p+2r}{q} - 2r < 0$, we have

$$0 < q\left(1 - b^p\right)^2 \le 0$$

by letting $a \to \infty$. This is a contradiction.

Step 2. (I) $\frac{1}{2} \le p \le 1$, (II) $-2r < p \le 1$.

We show that if p, q, r satisfy

$$-\frac{1}{2} \le r < 0, -2r < p \le 1, 0 < q < \frac{p+2r}{1+2r},$$

then there exist invertible operators $A, B \in B(\mathbf{R}^2)$ with $O \le B \le A$ which do not satisfy the inequality (1). (We remark that, in the condition (I), if $-\frac{1}{4} < r < 0$, then $\{p \in \mathbf{R} \mid \frac{1}{2} \le p \le 1\} \subset \{p \in \mathbf{R} \mid -2r < p \le 1\}$, and if $r = -\frac{1}{4}, p = \frac{1}{2}$, then the inequality (1) is valid for all $q > 0$.) We remark that if the inequality (1) is valid for all invertible operators $A, B \in B(H)$ satisfying $O \le B \le A$, then the inequality (1) is valid for all operators $A, B \in B(H)$ satisfying $O \le B \le A$.

Let

$$A = \begin{pmatrix} 2 & 2\sqrt{c(1-c)} \\ 2\sqrt{c(1-c)} & 4c \end{pmatrix},$$

where $0 < c < 1$ and let

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $O \le B \le A$. To show the assertion by contradiction, we assume the inequality (1). Let

$$V = \begin{pmatrix} \sqrt{1-c} & \sqrt{c} \\ \sqrt{c} & -\sqrt{1-c} \end{pmatrix}.$$

Then V is unitary and

$$V^* A V = \begin{pmatrix} 2 + 2c & 0 \\ 0 & 2c \end{pmatrix}.$$

By (1), we have

$$\left((V^*AV)^r\, V^*B^pV\, (V^*AV)^r\right)^{\frac{1}{q}} \le (V^*AV)^{\frac{p+2r}{q}}$$

and

$$\begin{aligned}&\begin{pmatrix} (2+2c)^{2r}(1-c) & (2+2c)^r(2c)^r\sqrt{c(1-c)} \\ (2+2c)^r(2c)^r\sqrt{c(1-c)} & (2c)^{2r}c \end{pmatrix}^{\frac{1}{q}} \\ &\quad\le \begin{pmatrix} (2+2c)^{\frac{p+2r}{q}} & 0 \\ 0 & (2c)^{\frac{p+2r}{q}} \end{pmatrix}\end{aligned}$$

and, by Lemma 5, we have

$$\begin{aligned}&\delta^{-1}\begin{pmatrix} (2+2c)^{2r}(1-c) & (2+2c)^r(2c)^r\sqrt{c(1-c)} \\ (2+2c)^r(2c)^r\sqrt{c(1-c)} & (2c)^{2r}c \end{pmatrix} \\ &\quad\le \begin{pmatrix} (2+2c)^{\frac{p+2r}{q}} & 0 \\ 0 & (2c)^{\frac{p+2r}{q}} \end{pmatrix}\end{aligned}$$

and hence

$$0 \le \begin{pmatrix} \delta(2+2c)^{\frac{p+2r}{q}} - (2+2c)^{2r}(1-c) & -(2+2c)^r(2c)^r\sqrt{c(1-c)} \\ -(2+2c)^r(2c)^r\sqrt{c(1-c)} & \delta(2c)^{\frac{p+2r}{q}} - (2c)^{2r}c \end{pmatrix},$$

where

$$\delta^{-1} = \left((2+2c)^{2r}(1-c) + (2c)^{2r}c\right)^{\frac{1}{q}-1}.$$

Therefore

$$0 \le \delta^2(2+2c)^{\frac{p+2r}{q}}(2c)^{\frac{p+2r}{q}} - \delta(2c)^{\frac{p+2r}{q}}(2+2c)^{2r}(1-c) - \delta(2+2c)^{\frac{p+2r}{q}}(2c)^{2r}c,$$

$$(2c)^{\frac{p+2r}{q}}(2+2c)^{2r}(1-c) + (2+2c)^{\frac{p+2r}{q}}(2c)^{2r}c \le \delta(2+2c)^{\frac{p+2r}{q}}(2c)^{\frac{p+2r}{q}}$$

and

$$\begin{aligned}&2^{\frac{p+2r}{q}}(2+2c)^{2r}(1-c)c^{\frac{p+2r}{q}-1-2r} + (2+2c)^{\frac{p+2r}{q}}2^{2r} \\ &\quad\le \left\{(2+2c)^{2r}(1-c) + 2^{2r}c^{1+2r}\right\}^{1-\frac{1}{q}}(2+2c)^{\frac{p+2r}{q}}2^{\frac{p+2r}{q}}c^{\frac{p+2r}{q}-1-2r}.\end{aligned}$$

Since $0 < \frac{p+2r}{q} - 1 - 2r$, we have

$$0 < 2^{\frac{p+2r}{q}}2^{2r} \le 0$$

by letting $c \to +0$. This is a contradiction. (We remark that step 2 is also valid for the condition (I) with $-2r < p \le 1$.)

Step 3. (II) $\frac{1}{2} \le p < -2r$, (III) $\frac{1}{2} \le p < 1$.

We show that if p, q, r satisfy

$$-\frac{1}{2} \le r < -\frac{1}{4}, \frac{1}{2} \le p < -2r, 0 < q < \frac{p+2r}{2p+2r-1},$$

then there exist invertible operators $A, B \in B(\mathbf{R}^2)$ with $O \le B \le A$ which do not satisfy the inequality (1).

Define A, B as in step 1. To show the assertion by contradiction, we assume the inequality (1). Similarly as in step 1, we can show the inequality (7). Since $0 < -\frac{p+2r}{q} + 2p + 2r - 1$, we have

$$0 < q\left(1 - a^{-1}\right) \leq 0$$

by letting $b \to +0$ in (7). This is a contradiction.

Step 4. (I) $1 < p$, (II) $1 < p$, (III) $1 < p$.

We show that if p, q, r satisfy

$$-\frac{1}{2} \leq r < 0, 1 < p, 0 < q,$$

then there exist invertible operators $A, B \in B(\mathbf{R}^2)$ with $O \leq B \leq A$ which do not satisfy the inequality

$$B^{\frac{p+2r}{q}} \leq (B^r A^p B^r)^{\frac{1}{q}}, \tag{3}$$

or, equivalently,

$$I \geq B^{\frac{p+2r}{2q}} \left(B^{-r} A^{-p} B^{-r}\right)^{\frac{1}{q}} B^{\frac{p+2r}{2q}}. \tag{8}$$

We remark that if the inequality (8) is valid for all invertible operators $A, B \in B(H)$ satisfying $O \leq B \leq A$, then the inequality (8) is valid for all operators $A, B \in B(H)$ satisfying $O \leq B \leq A$. Define A, B, V as in step 2. To show the assertion by contradiction, we assume the inequality (8). Since

$$I \geq B^{\frac{p+2r}{2q}} \left(B^{-r} V V^* A^{-p} V V^* B^{-r}\right)^{\frac{1}{q}} B^{\frac{p+2r}{2q}},$$

we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \geq \left\{(1-c)(2+2c)^{-p} + c(2c)^{-p}\right\}^{\frac{1}{q}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence

$$1 \geq (1-c)(2+2c)^{-p} + c(2c)^{-p} = (1-c)(2+2c)^{-p} + 2^{-p} c^{1-p}.$$

Since $1 - p < 0$, we have $1 \geq \infty$ by letting $c \to +0$. This is a contradiction.

Step 5. (IV) $1 < p$.

We show that if p, q, r satisfy

$$1 < p, 0 < q, r < 0,$$

then there exist invertible operators $A, B \in B(\mathbf{R}^2)$ with $O \leq B \leq A$ which do not satisfy the inequality (1) or (2).

First, let $0 < p + 2r$. Then the proof is similar to step 2.

Second, let $p + 2r = 0$. Then

$$\begin{aligned} (A^r B^p A^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}} &\iff \left(A^r B^{-2r} A^r\right)^{\frac{1}{q}} \leq A^0 = I \\ &\iff A^r B^{-2r} A^r \leq I \\ &\iff B^{-2r} \leq A^{-2r}. \end{aligned}$$

The last inequality is valid if and only if $-1 \leq 2r \leq 0$.

Last, let $p + 2r < 0$. We remark that the inequality (2) is equivalent to

$$B^{\frac{-p-2r}{q}} \geq \left(B^{-r} A^{-p} B^{-r}\right)^{\frac{1}{q}}. \tag{9}$$

Define A, B, V as in step 2. To show the assertion by contradiction, we assume the inequality (9). Since

$$B^{\frac{-p-2r}{q}} \geq \left(B^{-r}VV^*A^{-p}VV^* = B^{-r}\right)^{\frac{1}{q}},$$

we have

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \geq \left\{(1-c)(2+2c)^{-p} + c(2c)^{-p}\right\}^{\frac{1}{q}} \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

The rest of the proof is similar to step 4.

Step 6. (IV) $0 < p \leq 1$.

We show that if p, q, r satisfy

$$0 < p \leq 1, 0 < q, r < 0,$$

then there exist invertible operators $A, B \in B(\mathbf{R}^2)$ with $O \leq B \leq A$ which do not satisfy the inequality (1).

Define A, B as in step 2. To show the assertion by contradiction, we assume the inequality (1). Then, similarly as in step 2, we have

$$\begin{aligned}
&2^{\frac{p+2r}{q}}(2+2c)^{2r}(1-c) + (2+2c)^{\frac{p+2r}{q}}2^{2r}c^{1+2r-\frac{p+2r}{q}}\\
&\leq \left\{(2+2c)^{2r}(1-c) + 2^{2r}c^{1+2r}\right\}^{1-\frac{1}{q}}(2+2c)^{\frac{p+2r}{q}}2^{\frac{p+2r}{q}}\\
&= \left\{(2+2c)^{2r}(1-c)c^{-1-2r} + 2^{2r}\right\}^{1-\frac{1}{q}}c^{(1+2r)(1-\frac{1}{q})}(2+2c)^{\frac{p+2r}{q}}2^{\frac{p+2r}{q}}.
\end{aligned}$$

Since

$$(1+2r)\left(1-\frac{1}{q}\right) < 0 < 1 + 2r - \frac{p+2r}{q},$$

we have

$$0 < 2^{\frac{p+2r}{q}}2^{2r} \leq 0$$

by letting $c \to +0$. This is a contradiction.

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