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BINOMIAL FREE RESOLUTIONS FOR NORMAL TORIC SURFACES

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ABSTRACT. We construct the minimal free resolution of the residue field over a normal toric surface.

I. INTRODUCTION

Any normal toric surface is given by a normal 2-dimensional submonoid Λ of $\mathbb{N}^2$ (cf. [Fu]). Denote by $\alpha_1, \dots, \alpha_n$ the minimal generators of Λ. Then

$$k[\Lambda] \cong k[x_1, \dots, x_n]/I_\Lambda,$$

where I_Λ is the *toric ideal* equal to the kernel of the map $k[x_1, \dots, x_n] \to k[z_1, z_2]$ sending x_i to $\mathbf{z}^{\alpha_i} = z_1^{\alpha_{i1}} z_2^{\alpha_{i2}}$. The ideal I_Λ might not be homogeneous with respect to the usual grading, but it is always $\mathbb{N}^2$-graded.

We are interested in resolving k as a $k[\Lambda]$-module. The Betti numbers were first studied in [LS], and then it was proved in [PRS] (also see [HRW]) that

$$\dim_k \mathrm{Tor}_i^{k[\Lambda]}(k,k) = (n-2)^{i-2}(n-1)^2 \qquad \text{for } i \geq 2.$$

In this paper we construct a minimal free resolution $\mathbb{F}$ of the residue field k over $k[\Lambda]$. The resolution has the following properties:

- the basis elements of $\mathbb{F}$ as a $k[\Lambda]$-module and the differential maps are given explicitly by simple formulas;
- the resolution $\mathbb{F}$ is binomial: for $i \geq 2$ denote by D_i the matrix of the differential d_i in $\mathbb{F}$, then any column in D_i has two non-zero entries and they are monomials;
- $\mathbb{F}$ lifts the minimal free resolution in [Fr, Theorem in §3] of k over $k[x_1, \dots, x_n]/in_\prec(I_\Lambda)$, where $in_\prec(I_\Lambda)$ is a quadratic initial ideal.

The construction of $\mathbb{F}$ is presented in Section 2 and is proved in Section 3. We demonstrate the construction in Example 2.5 for the twisted cubic curve; in this very special example minimal free resolutions of k are given by Golod's construction, Priddy's construction, and Anick's construction, however none of them is as simple as $\mathbb{F}$. The motivation for this paper was to obtain explicit nice minimal free resolutions for all rational normal curves.

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II. Minimal free resolution

In this section we construct the minimal free resolution of k over a normal toric surface and demonstrate the construction in Example 2.5.

Order the unique set of minimal generators $\alpha_1,\dots,\alpha_n$ of Λ so that $\det(\alpha_i,\alpha_{i+1}) \le 0$ for $i = 1, 2,\dots, n-1$. Let $\prec$ be the purely lexicographic term order on $k[x_1,\dots,x_n]$. The toric ideal I_Λ has a minimal Gröbner basis consisting of the $\binom{n-1}{2}$ binomials:

(2.1)
$$\underline{x_i x_{i+j}} - x_{i+1}x_{i+j-1}\prod_{p=i+1}^{i+j-1} x_p^{b_p}, \qquad 1 \le i \le n-2,\ 2 \le j \le n-i.$$

(For definition and properties of Gröbner basis see [Ei].) The underlined monomials generate the initial ideal $in_\prec(I_\Lambda) = \big(x_i x_j \,:\, 1 \le i < j-1 \le n-1\big)$, which is the Stanley-Reisner ideal for the "zig-zag poset" poset P on $\{x_1, x_2,\dots,x_n\}$ (namely, P has covering relations $x_i <_P x_{i+1}$ if i is odd and $x_i >_P x_{i+1}$ if i is even).

A minimal free resolution of k over $k[x_1,\dots,x_n]/in_\prec(I_\Lambda)$ is given in [Fr, Theorem in §3]. Our construction lifts this resolution to $k[\Lambda]$ using the specific relations (2.1).

We now construct a minimal free resolution $\mathbb{F}$ of k over $k[\Lambda]$. We define $\mathbb{F}$ to be the free $k[\Lambda]$-module $k[\Lambda] \otimes_k Q$, where Q is the non-commutative k-algebra

$$Q = k\langle y_1,\dots,y_n\rangle / \big(\{ y_i y_{i+1} + y_{i+1} y_i \}_{1\le i\le n-1},\ \{ y_j^2 \}_{1\le j\le n}\big).$$

As we mentioned in the introduction, by [PRS] the *Betti numbers* of k are

$$\dim_k \operatorname{Tor}_i^{k[\Lambda]}(k,k) = (n-2)^{i-2}(n-1)^2 \qquad \text{for } i \ge 2.$$

So the *Poincaré series* of k is

$$\frac{(1+t)^2}{1-(n-2)t}$$

and it can be seen by [Fr] that the Hilbert series of Q is the same. We say that a y-monomial $y_{i_1} y_{i_2} \dots y_{i_p} \in Q$ is *standard* if y_{i_1} is the smallest variable that can be factored to the left (modulo the relations defining Q) and $y_{i_2} \dots y_{i_p}$ is standard. The standard y-monomials of degree i form a basis for the free $k[\Lambda]$-module $\mathbb{F}_i$. In what follows the letter m denotes a standard y-monomial.

Next we define a differential d on $\mathbb{F}$ which is homogeneous with respect to the $\mathbb{N}^2$-grading. For y-monomials of degree ≤ 2 we set

$$\begin{aligned} d(y_s) &= x_s, \\ d(y_i y_{i+1}) &= -d(y_{i+1} y_i) = x_i\, y_{i+1} - x_{i+1}\, y_i, \\ d(y_i y_{i+j}) &= x_i\, y_{i+j} - c_{ij}\, y_{i+j-1}, \\ d(y_{i+j} y_i) &= x_{i+j}\, y_i - c'_{ij}\, y_{i+1}, \end{aligned} \tag{2.2}$$

where $1 \le s \le n$, $1 \le i \le n-1$, and $2 \le j \le n-i$. The coefficients $c_{ij} = x_{i+1}\prod_{p=i+1}^{i+j-1} x_p^{b_p}$ and $c'_{ij} = x_{i+j-1}\prod_{p=i+1}^{i+j-1} x_p^{b_p}$ are uniquely determined by (2.1); this choice of the coefficients ensures that the differential is homogeneous with respect to the $\mathbb{N}^2$-grading.

Let $y_i y_j y_l\, m$ be an arbitrary standard y-monomial of degree ≥ 3. We define

$$d(y_i y_j y_l\, m) = \begin{cases} d(y_i y_l)\, y_j\, m & \text{if } i > l = j+1 \\ d(y_i y_j)\, y_l\, m & \text{otherwise.} \end{cases} \tag{2.3}$$

Finally we extend the action of the differential d to all of $\mathbb{F}$ by $k[\Lambda]$-linearity. The following theorem is our main result; it is proved in the next section.

Theorem 1. *$(\mathbb{F}, d)$ is a minimal free resolution of k over $k[\Lambda]$.*

Example 2.5. The twisted cubic curve. We consider the monoid Λ generated by $\alpha_1 = (0,3), \alpha_2 = (1,2), \alpha_3 = (2,1), \alpha_4 = (3,0) \in \mathbb{N}^2$. The monoid algebra

$$k[\Lambda] \cong k[x_1, \dots, x_4]/\big(x_1x_3 - x_2^2, x_2x_4 - x_3^2, x_1x_4 - x_2x_3\big)$$

is the toric ring of the twisted cubic curve. In this case the equations in (2.1) are the defining equations. The minimal free resolution of k over $k[\Lambda]$ is

$$\mathbb{F} = k[\Lambda] \otimes k\langle y_1, \dots, y_4\rangle / \big(\left\{ y_iy_{i+1} + y_{i+1}y_i \right\}_{1\le i\le 3}, \left\{ y_i^2 \right\}_{1\le i\le 4} \big).$$

The differential acts on the standard monomials in the following way (the notation is as in (2.2) and (2.3)):

$$d(y_i) = x_i,$$
$$d(y_iy_{i+1}) = x_iy_{i+1} - x_{i+1}y_i,$$

$$d(y_1y_3) = x_1y_3 - x_2y_2, \qquad d(y_2y_4) = x_2y_4 - x_3y_3, \qquad d(y_1y_4) = x_1y_4 - x_2y_3,$$
$$d(y_3y_1) = x_3y_1 - x_2y_2, \qquad d(y_4y_2) = x_4y_2 - x_3y_3, \qquad d(y_4y_1) = x_4y_1 - x_3y_2,$$

$$\begin{aligned} d(y_3y_1y_2m) &= d(y_3y_2)y_1m, \\ d(y_4y_1y_2m) &= d(y_4y_2)y_1m, \\ d(y_4y_2y_3m) &= d(y_4y_3)y_2m, \\ d(y_iy_jy_lm) &= d(y_iy_j)y_lm \quad \text{otherwise.} \end{aligned}$$

III. Proof

In this section we prove Theorem 2.4 in a sequence of three lemmas:

Theorem 2. *$(\mathbb{F}, d)$ is a complex.*

Proof. In order to simplify the notation we consider $(\bar{\mathbb{F}}, \bar{d}) = (\mathbb{F}, d)/\big(\left\{x_i - 1\right\}_{1\le i\le n} \big)$. Since $(\mathbb{F}, d)$ is multigraded and $k[\Lambda]$ is one-dimensional (over k) in each multidegree, it suffices to show that $\bar{d}^2$ annihilates all standard y-monomials. The action of $\bar{d}$ on a standard y-monomial M depends only on the arrangement of the three leftmost variables in M. Our proof splits into seven possible cases for these arrangements.

Case 1. If M has degree ≤ 2, then $\bar{d}^2(M) = 0$ by (2.2) and the relations (2.1).

In the remaining cases $\deg(M) \ge 3$ and i, j, q, p are suitable positive integers.

Case 2. Let $M = y_iy_{i+j}y_{i+j+q}m$. Then

$$\begin{aligned} \bar{d}^2(M) &= \bar{d}(y_{i+j}y_{i+j+q}m - y_{i+j-1}y_{i+j+q}m) \\ &= y_{i+j+q}m - y_{i+j+q-1}m - y_{i+j+q}m + y_{i+j+q-1}m = 0. \end{aligned}$$

Case 3. Let $M = y_iy_{i+j}y_{i+j-2}m$. Note that y_{i+j-1} cannot be the first variable of m since it commutes with $y_{i+j}y_{i+j-2}$ and M is standard. Hence

$$\begin{aligned} \bar{d}^2(M) &= \bar{d}(y_{i+j}y_{i+j-2}m - y_{i+j-1}y_{i+j-2}m) \\ &= y_{i+j-2}m - y_{i+j-1}m + y_{i+j-1}m - y_{i+j-2}m = 0. \end{aligned}$$

Case 4. Let $M = y_i y_{i+j} y_{i+j-p} m$, where $p \geq 3$. Then

$$\bar{d}^2(M) = \bar{d}(y_{i+j} y_{i+j-p} m - y_{i+j-1} y_{i+j-p} m).$$

This evaluates to $y_{i+j-p} m - y_{i+j-p+1} m - y_{i+j-p} m + y_{i+j-p+1} m = 0$ if m does not start with $y_{i+j-p+1}$. Otherwise we write $m = y_{i+j-p+1} \tilde{m}$ and

$$\bar{d}^2(M) = \bar{d}(y_{i+j} y_{i+j-p+1}) y_{i+j-p} \tilde{m} - \bar{d}(y_{i+j-1} y_{i+j-p+1}) y_{i+j-p} \tilde{m} = 0.$$

Case 5. Let $M = y_{i+j} y_i y_{i+q} m$, where $q \geq 2$. Then

$$\begin{aligned} \bar{d}^2(M) &= \bar{d}(y_i y_{i+q} m - y_{i+1} y_{i+q} m) \\ &= y_{i+q} m - y_{i+q-1} m - y_{i+q} m + y_{i+q-1} m = 0. \end{aligned}$$

Case 6. Let $M = y_{i+j} y_i y_{i+1} m$, where $j \geq 2$. Here m cannot start with y_{i+1}; therefore

$$\begin{aligned} \bar{d}^2(M) &= \bar{d}\big(\bar{d}(y_{i+j} y_{i+1}) y_i m)\big) \\ &= \bar{d}\big(y_{i+1} y_i m - y_{i+2} y_i m\big) \\ &= y_i m - y_{i+1} m - y_i m + y_{i+1} m = 0. \end{aligned}$$

Case 7. Let $M = y_{i+j} y_i y_{i-p} m$, where $p \geq 2$. Note that if $p = 2$, then m cannot start with y_{i-1} because y_{i-1} commutes with $y_i y_{i-2}$ and M is standard. We have $\bar{d}^2(M) = \bar{d}(y_i y_{i-p} m - y_{i+1} y_{i-p} m)$. This evaluates to $y_{i-p} m - y_{i-p+1} m - y_{i-p} m + y_{i-p+1} m = 0$ if m does not start with y_{i-p+1}. Otherwise write $m = y_{i-p+1} \tilde{m}$ and

$$\bar{d}^2(M) = \bar{d}(y_i y_{i-p+1})\, y_{i-p} \tilde{m} - \bar{d}(y_{i+1} y_{i-p+1})\, y_{i-p} \tilde{m} = 0.$$

□

The complex $\mathbb{F}$ is spanned as a k-vector space by monomials in x and y. The commutative x-part obeys the relations in $k[\Lambda]$ and the non-commutative y-part obeys the relations in Q. We call such a monomial *spanning* if it has the form $x_i^c x_{i+1}^e y_{i+1} y_j m$ where either $y_{i+1} y_j m$ is standard or $i = j$ and $y_i y_{i+1} m$ is standard.

Theorem 3. *The set of spanning monomials spans the k-vector space $\mathbb{F}/\mathrm{Im}(d)$.*

Proof. Every monomial in $\mathbb{F}_{\geq 2}$ can be written in the form $M = x_i^c x_{i+1}^e y_r y_s m$, where $y_r y_s m$ is standard. If $c = e = 0$, then M is spanning. We therefore assume $c > 0$. We set $X(M) = i$ and $Y(M) = r$. Thus M is spanning if and only if $Y(M) - X(M) = 1$. We will show that every non-spanning monomial M is congruent modulo $\mathrm{Im}(d)$ to another monomial M' which is closer to being spanning. Depending on whether $X(M) - Y(M)$ is positive, zero, or negative, we use one of the following reduction procedures:
Negative reduction: Suppose $M = x_i^w x_{i+1}^v y_{i+j} y_z m$ with $j \geq 2$. Set

$$M' = M - d(x_i^{w-1} x_{i+1}^v y_i y_{i+j} y_z m).$$

Rewriting M' in standard form, we see that $Y(M) > Y(M')$ and $X(M) \leq X(M')$. Thus M' is congruent to M modulo $\mathrm{Im}(d)$ and $Y(M) - X(M) > Y(M') - X(M')$.
Positive reduction 1: Suppose $M = x_{i+j}^w x_{i+j+1}^v y_i y_z m$, $z \neq i+1$ and $j > 0$. Set $M' = M - d(x_{i+j}^{w-1} x_{i+j+1}^v y_{i+j} y_i y_z m)$. Rewriting M' in standard form, we find $Y(M) < Y(M')$ and $X(M) \geq X(M')$, hence $X(M) - Y(M) > X(M') - Y(M')$.

Positive reduction 2: Suppose $M = x_{i+j}^w x_{i+j+1}^v y_i y_{i+1} m$ with $j > 0$. Set $M' = -M - d(x_{i+j}^{w-1} x_{i+j+1}^v y_{i+j} y_{i+1} y_i m)$. Then $X(M) - Y(M) > X(M') - Y(M')$.
Zero reduction: Suppose $M = x_i^w x_{i+1}^v y_i y_z m$, where $v > 0$ and $z \neq i+1$. Then

$$M' = d(x_i^w x_{i+1}^{v-1} y_i y_{i+1} y_z m) + M \ = \ x_i^{w+1} x_{i+1}^{v-1} y_{i+1} y_z m$$

is a spanning monomial.

Starting with any monomial M in $\mathbb{F}_{\geq 2}$, we can use a sequence of reductions as above to replace M by a spanning monomial modulo $\mathrm{Im}(d)$. □

Theorem 4. *The complex* $(\mathbb{F}, d)$ *is exact.*

Proof. Consider any non-zero Λ-homogeneous k-linear combination of spanning monomials,

$$N \ = \sum_s \ \beta_s x_{q_s}^{w_s} x_{q_s+1}^{v_s} y_{q_s+1} y_{z_s} m_s \ ,$$

where $\beta_s \in k$. To prove Lemma 3.3, we must show that $d(N) \neq 0$. We shall assume that each monomial in N is written so that the differential acts on the first two y-variables (to obtain this we switch the second and third y-variables if necessary). We call z_s the *order* of the term $x_{q_s}^{w_s} x_{q_s+1}^{v_s} y_{q_s+1} y_{z_s} m_s$. Setting $u = \min\{z_s\}$, we can write

$$N \ = \ \beta x_q^w x_{q+1}^v y_{q+1} y_u m \ + \ \text{terms of order} \geq u \ ,$$

where $\beta \in k$. Since $k[\Lambda]$ is one-dimensional in each multidegree, N is multihomogeneous, and each term of N is spanning, we find that there is only one term in N which has order u and ends on m.

Suppose that $d(N) = 0$. The term $L = c x_q^w x_{q+1}^{v+1} y_u m$ appears in $d(N)$ and must cancel. Since N contains no spanning monomials of order $u-1$, a scalar multiple of L must appear in $d(x_h^f x_{h+1}^g y_{h+1} y_{u+1} m)$ for some f, g, h. This is only possible if $u > q \geq h$, and in view of $y_{q+1} y_u \neq 0$, we conclude that $u \geq q+2$. But then the spanning monomial $c x_q^w x_{q+1}^v y_{q+1} y_u m$ contributes a term $p y_{u-1} m$ (with p a monomial in the x-variables) to the expansion of $d(N) = 0$. This term $p y_{u-1} m$ appears in the differential of another spanning monomial from N. Such a spanning monomial must have order u and end on m. Hence N contains two or more distinct spanning monomials of order u and ending on m. This is a contradiction. Thus, $(\mathbb{F}, d)$ is exact. □

This completes the proof of Theorem 2.4 since $(\mathbb{F}, d)$ is minimal by construction (the entries in the matrices of the differential maps are in the ideal $(x_1, \ldots, x_n)$).

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