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THE UNILATERAL SHIFT AND A NORM EQUALITY FOR BOUNDED LINEAR OPERATORS

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Dedicated to Professor Yenn Tseng on her retirement.

ABSTRACT. The paper gives a necessary and sufficient condition for the norm equality $\| S-T \|=\| S \| + \| T \|$ of bounded linear operators S and T. The invertibility of an operator which is related to the norm equality is discussed. Some new results about the unilateral shift are given.

One of the consequences from the condition considered in the abstract is that if U is the unilateral shift and V is any unitary operator on a Hilbert space, then the following are equivalent: (1) $\| U-V \|= 2$; (2) 0 is in the approximate point spectrum of the operator $U+V$; (3) the operator $U+V$ is noninvertible; and (4) $\| cU-(1+c)V \|= 1+2c$ for any positive real number c. Moreover, each statement in above holds true.

Despite the fact that the approximate point spectrum of the direct sum of two operators is the union of their approximate point spectra [2, p. 50], the situation with sums of different non-scalar operators is somewhat more complicated, and apparently very little is known in the literature. What is peculiar about the result stated above is that the approximate point spectrum of U is precisely the unit circle, while that of V is contained in the unit circle.

Let $B(X)$ denote the algebra of all bounded linear operators on a specified space X, and I the identity operator. We first cite a lemma, the proof of which is trivial and well known; e.g., [1, Lemma 2.1].

Lemma 1. *Let x and y be vectors in a normed space such that $\| x+y \|=\| x \| + \| y \|$, and let a and b be nonnegative real numbers. Then $\| ax+by \|= a \| x \| +b \| y \|$.*

Theorem 1. *Let $S, T \in B(X)$, where X is a uniformly convex Banach space. If $\| S-T \|=\| S \| + \| T \|$, then 0 is in the approximate point spectrum of the operator $\| T \| S+ \| S \| T$. The converse holds if X is a Hilbert space and if any one of S and T is an isometric operator.*

Proof. Suppose that both S and $T \neq O$, $\| S-T \|=\| S \| + \| T \|$, and write $P=\frac{S}{\|S\|}$ and $Q=\frac{-T}{\|T\|}$. Since $\| S-T \|=\| S \| + \| T \|$, we have

$$\| P+Q \|= 2$$

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by Lemma 1. So, there exists a sequence $\{x_n\}$ of unit vectors such that

$$\lim_{n\to\infty} \| Px_n + Qx_n \| = 2,$$

which implies that

$$\lim_{n\to\infty} \| Px_n - Qx_n \| = 0$$

as X is uniformly convex, $\| Px_n \| \leq 1$, and $\| Qx_n \| \leq 1$ for all n. It follows that $\lim_{n\to\infty} \| (\| T \| S + \| S \| T)x_n \| = 0$.

Conversely, let X be a Hilbert space, let S be an isometric operator (similarly for T), and let $\lim_{n\to\infty} \| (\| T \| S + T)x_n \| = 0$ for some sequence $\{x_n\}$ of unit vectors. Then we have

$$\begin{aligned} 2 \| T \| &\geq \| (\| T \| S - T)x_n \| \\ &\geq 2 \| T \| \| Sx_n \| - \| (\| T \| S + T)x_n \| \\ &\to 2 \| T \| \text{ as } n \to \infty. \end{aligned}$$

Hence, $\| \| T \| S - T \| = 2 \| T \|$, equivalently, $\| S - T \| = 1 + \| T \|$ by Lemma 1.

Consequently, we have the next result which will be used frequently later.

Theorem 2. *Let S, $T \in B(X)$, where X is a Hilbert space and S is an isometric operator. Then $\| S - T \| = 1 + \| T \|$ if and only if 0 is in the approximate point spectrum of the operator $\| T \| S + T$.*

Let us consider the invertibility of an operator. It is standard that if $T \in B(X)$, where X is a Banach space, and if $\| I - T \| < 1$, then T is invertible. The converse does not hold in general, but it does if in addition $\| I - rT \| = 1$ for some real number $r > 1$ [3, Theorem 2]. If X is a Hilbert space, the identity operator may be replaced by any unitary operator as the next result shows.

Corollary 1. *Let V, $T \in B(X)$, where X is a Hilbert space, and V is any unitary operator. If $\| V - T \| < 1$, then T is invertible. The converse holds if $\| V - rT \| = 1$ for some real number $r > 1$.*

Proof. Since $\| I - V^*T \| = \| V - T \| < 1$, V^*T is invertible. Hence, T is invertible as is V^*. Conversely, let T be invertible and $\| V - rT \| = 1$ for some real number $r > 1$. Due to Lemma 1, Theorem 2 may be rephrased as follows: $\| (r-1)S - T \| < (r-1) + \| T \|$ if and only if 0 is not in the approximate point spectrum of the operator $\| T \| S + T$. Thus, in our case 0 is not in the approximate point spectrum of the operator $rT = \| V - rT \| V + (rT - V)$, if and only if $\| (r-1)V - (rT - V) \| < (r-1) + \| rT - V \| = r$. Hence,

$$r \| V - T \| = \| rV - rT \| = \| (r-1)V - (rT - V) \| < r,$$

and so, $\| V - T \| < 1$.

By Corollary 1 we may characterize Theorem 2 in terms of noninvertibility of an operator as follows.

Corollary 2. *Let V, $T \in B(X)$, where X is a Hilbert space, and V is any unitary operator. Then $\| V - T \| = 1 + \| T \|$ if and only if the operator $\| T \| V + T$ is noninvertible.*

Proof. In Corollary 1 let $r > 1 = \| T \|$ and $W = \frac{V+T}{r}$, then $T = rW - V$ and $\| V - rW \| = \| T \| = 1$. Thus, $W = \frac{V+T}{r}$ is noninvertible if and only if $\| V - \frac{V+T}{r} \| \geq 1$, i.e., $\| (r-1)V - T \| = r$. Equivalently, $\| V - T \| = 1 + \| T \|$ by Lemma 1.

Theorem 3. *Let X be a Hilbert space, and $U, V \in B(X)$, where U is the unilateral shift and V is any unitary operator. Then the following are equivalent:*

(1) $\| U - V \| = 2$;
(2) 0 *is in the approximate point spectrum of the operator* $U + V$;
(3) *the operator* $U + V$ *is noninvertible;*
(4) $\| cU - (1+c)V \| = 1 + 2c$ *for any positive real number* c.

Moreover, each statement above holds true.

Proof. (1) holds true since it is well known that the distance between the unilateral shift and any unitary operator is 2 [2, Solution 119, p. 275], i.e.,

$$2 = \| U - V \| \leq \| U \| + \| V \| = 1 + 1 = 2.$$

And so, the equivalence of (1), (2), and (3) follows from Theorem 2 and Corollary 2. Now, for any positive real number c let $T = c(U - V)$ in Corollary 2, then $\| T \| = 2c$. Accordingly, the operator $c(U + V) = 2cV + c(U - V)$ is noninvertible if and only if $\| cU - (1+c)V \| = \| c(U - V) - V \| = 1 + 2c$. This shows that (3) and (4) are equivalent.

Some interesting properties about the unilateral shift are indicated in the next result, which is a special case of Theorem 3 with a unitary operator V replaced by a complex number of modulus 1.

Corollary 3. *If U is the unilateral shift on a Hilbert space, and if γ is a complex number of modulus 1, then the following are equivalent:*

(1) $\| U - \gamma I \| = 2$;
(2) 0 *is in the approximate point spectrum of the operator* $U + \gamma I$;
(3) *The operator* $U + \gamma I$ *is noninvertible;*
(4) $\| cU - (1+c)\gamma I \| = 1 + 2c$ *for any positive real number* c.

Moreover, each statement above holds true.

Remarks. (a) Let S, $T \in B(X)$, where X is a uniformly convex Banach space. It can be shown that if $\| S - T \| = \alpha$, and both $\| S \|$and $\| T \| \leq \frac{\alpha}{2}$, so that $\| S - T \| = \| S \| + \| T \| = \alpha$, then 0 is in the approximate point spectrum of the operator $S + T$. On the other hand, if X is merely a normed space, and if $\| S \| + \| T \|$ is in the approximate point spectrum of the operator $S - T$, then it can be shown that $\| S - T \| = \| S \| + \| T \|$.

(b) In Corollary 3 let U be the unilateral shift acting on the space $l_2(\mathbf{z})$ in particular. Easy applications of the inner product in $l_2(\mathbf{z})$ and of the spectral mapping theorem show that the same four conditions for U are eqivalent and true.

(c) Let S, $T \in B(X)$, where X is a Hilbert space. For some sequence $\{x_n\}$ of unit vectors the following are equivalent:

(1) $\| I + S - T \| = 1 + \| S \| + \| T \|$;
(2) $\lim_{n\to\infty} \| (S - T)x_n - (\| S \| + \| T \|)x_n \| = 0$;
(3) $\lim_{n\to\infty} \| Sx_n - \| S \| x_n \| = \lim_{n\to\infty} \| Tx_n + \| T \| x_n \| = 0$;
(4) $\lim_{n\to\infty}(Sx_n \mid x_n) = \| S \|$ and $\lim_{n\to\infty}(Tx_n \mid x_n) = - \| T \|$;

(5) $\lim_{n\to\infty}((S-T)x_n \mid x_n) = \| S \| + \| T \|$;
(6) The operator $(S-T) - (\| S \| + \| T \|)I$ is noninvertible.

Moreover, each statement above implies that $\| S-T \| = \| S \| + \| T \|$.

We notice that the four equivalent statements (1), (2), (3), and (6) above are proved in [3, Theorem 4], and the others are easy exercises. The last statement follows from Remark (a).

References

[1] Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw, The Daugavet equation in uniformly convex Banach spaces, J. Funct. Anal. 97(1991), 215-230. MR **92i:**47005

[2] P.R. Halmos, A Hilbert Space Problem Book, Van Nostrand, Princeton, N.J., 1967. MR **34:**8178

[3] C.-S. Lin, Generalized Daugavet equations and invertible operators on uniformly convex Banach spaces, J. Math. Anal. and Appl. 197(1996), 518-528. MR **96m:**47069

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