

THE WEIERSTRASS APPROXIMATION THEOREM AND A CHARACTERIZATION OF THE UNIT CIRCLE

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ABSTRACT. We study real algebraic morphisms from nonsingular real algebraic varieties X with $\dim X \geq 1$ into nonsingular real algebraic curves C . We show, among other things, that the set of real algebraic morphisms from X into C is never dense in the space of all C^∞ maps from X into C , unless C is biregularly isomorphic to a Zariski open subset of the unit circle.

An *affine real algebraic variety* is a locally ringed space isomorphic to an algebraic subset of \mathbb{R}^n (for some n) endowed with the Zariski topology and the sheaf of \mathbb{R} -valued regular functions. For basic information on real algebraic varieties the reader may refer to [2]. Recall that every quasiprojective real algebraic variety is in fact affine [2, Proposition 3.2.10, Theorem 3.4.4]. Given two affine real algebraic varieties X and Y , we denote by $\mathcal{R}(X, Y)$ the set of all *regular maps* (that is, morphisms of locally ringed spaces) from X into Y . We assume that X and Y are nonsingular and regard $\mathcal{R}(X, Y)$ as a subset of the space $C^\infty(X, Y)$ of all C^∞ maps from X into Y endowed with the C^∞ compact-open topology (the weak C^∞ topology in the terminology used in [7]). It is natural to study the size of $\mathcal{R}(X, Y)$ in $C^\infty(X, Y)$ and several papers have already been devoted to this problem [2], [3], [4], [5]. In the present note we prove the following result, which can be viewed as a version, in a new setting, of the classical Weierstrass approximation theorem.

Theorem 1. *Given an affine nonsingular irreducible real algebraic curve C , the following conditions are equivalent:*

- (a) *The curve C is biregularly isomorphic to a Zariski open subset of the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$;*
- (b) *The set $\mathcal{R}(V, C)$ is dense in $C^\infty(V, C)$ for every compact affine nonsingular real algebraic curve V ;*
- (c) *There exists an affine nonsingular real algebraic variety X such that $\dim X \geq 1$ and $\mathcal{R}(X, C)$ is dense in $C^\infty(X, C)$.*

To prevent possible confusion let us say explicitly that throughout this note the adjective compact always refers to the Euclidean topology on the real algebraic varieties.

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Before giving a proof of Theorem 1 we make a few remarks. For every affine nonsingular irreducible real algebraic curve V there exists a unique (up to biregular isomorphism over \mathbb{R}) projective nonsingular irreducible complex algebraic curve $V_{\mathbb{C}}$ defined over \mathbb{R} such that V is biregularly isomorphic to a Zariski open subset of the set of real points $V_{\mathbb{C}}(\mathbb{R})$ of $V_{\mathbb{C}}$ (here $V_{\mathbb{C}}(\mathbb{R})$ is regarded as a projective real algebraic variety). We denote by $g(V)$ the genus of $V_{\mathbb{C}}$. Every regular map $f : V \rightarrow W$ of affine nonsingular irreducible real algebraic curves extends in a unique way to a (complex) regular map $f_{\mathbb{C}} : V_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$. Let us denote by $\mathcal{R}^*(V, W)$ the set of all nonconstant regular maps from V into W . By applying the remarks above and some classical theorems on projective complex algebraic curves we obtain information on the set $\mathcal{R}^*(V, W)$. Namely:

- (i) By the theorem of de Franchis [8, p. 227], if $g(W) \geq 2$, then the set $\mathcal{R}^*(V, W)$ is finite,
- (ii) The Hurwitz-Riemann formula [6, p. 140] implies that if $g(W) = 1$, then each map in $\mathcal{R}^*(V, W)$ has at most $2g(V) - 2$ critical points.

Proof of Theorem 1. Assume that (a) holds. It is known that the set $\mathcal{R}(V, S^1)$ is dense in $\mathcal{C}^\infty(V, S^1)$ for every compact affine nonsingular real algebraic curve V (cf. [4, Proposition 6.1, Theorem 1.6], where the problem of determining the size of $\mathcal{R}(X, S^1)$ in $\mathcal{C}^\infty(X, S^1)$ is completely solved for an arbitrary compact affine nonsingular real algebraic variety X). If C is biregularly isomorphic to a Zariski open subset of S^1 , different from S^1 , then C is biregularly isomorphic to a Zariski open subset of \mathbb{R} and hence the set $\mathcal{R}(X, C)$ is dense in $\mathcal{C}^\infty(X, C)$ for every compact affine nonsingular real algebraic variety X . Thus (b) is proved.

It is obvious that (b) implies (c). We now prove that (c) implies (a). Suppose that the set $\mathcal{R}(X, C)$ is dense in $\mathcal{C}^\infty(X, C)$ for some affine nonsingular real algebraic variety X with $\dim X \geq 1$. We shall show $g(C) = 0$, which is equivalent to (a). To this end let us choose a nonsingular irreducible real algebraic curve Z in X .

We have $g(C) \leq 1$ since otherwise the set $\mathcal{R}^*(Z, C)$ would be finite (cf. (i)) and therefore no map f in $\mathcal{C}^\infty(X, C)$ whose restriction $f|_Z$ is not in $\mathcal{R}(Z, C)$ could be approximated by regular maps from X into C .

Hence it remains to exclude the case $g(C) = 1$. Choose a \mathcal{C}^∞ map $h : X \rightarrow C$ such that $h|_Z$ has at least $2g(Z) - 1$ critical points, each of multiplicity exactly 2 (in particular, $h|_Z$ has in local coordinates either a local maximum or a local minimum at each critical point). If $g(C) = 1$, then it follows from (ii) that $h|_Z$ (and hence h) cannot be approximated by regular maps with values in C . Thus $g(C) = 0$ is proved. \square

The following characterization of the unit circle is an immediate consequence of Theorem 1.

Corollary 2. *Let C be a compact affine nonsingular irreducible real algebraic curve. Then the following conditions are equivalent:*

- (a) *The curve C is biregularly isomorphic to S^1 ;*
- (b) *The set $\mathcal{R}(V, C)$ is dense in $\mathcal{C}^\infty(V, C)$ for every compact affine nonsingular real algebraic curve V ;*
- (c) *There exists an affine nonsingular real algebraic variety X such that $\dim X \geq 1$ and $\mathcal{R}(X, C)$ is dense in $\mathcal{C}^\infty(X, C)$.* \square

It would be interesting, but probably hard, to find a generalization of Theorem 1 and Corollary 2 with C replaced by a higher dimensional real algebraic variety. In this direction we only have the following result.

Theorem 3. *Let M be a closed connected C^∞ manifold with $\dim M \geq 1$. Then there exists an affine nonsingular real algebraic variety Y such that M is diffeomorphic to Y and the set $\mathcal{R}(X, Y)$ is never dense in $C^\infty(X, Y)$, where X is an arbitrary affine nonsingular real algebraic variety with $\dim X \geq 1$.*

We shall need the following observation.

Lemma 4. *Let Y be an affine nonsingular irreducible real algebraic variety. Assume that there exists a nonconstant regular map $\varphi : Y \rightarrow C$ into an affine nonsingular irreducible real algebraic curve C with $g(C) \geq 2$. Then the set $\mathcal{R}(X, Y)$ is never dense in $C^\infty(X, Y)$, where X is an arbitrary affine nonsingular real algebraic variety with $\dim X \geq 1$.*

Proof. Since Y is irreducible and $\varphi : Y \rightarrow C$ is nonconstant, we can choose a sequence $\{U_n\}$ of nonempty open (in the Euclidean topology) subsets of Y such that the subsets $\varphi(U_n)$ of C are pairwise disjoint.

Suppose now that $\mathcal{R}(X, Y)$ is dense in $C^\infty(X, Y)$ for some affine nonsingular real algebraic set X with $\dim X \geq 1$. Let Z be a nonsingular irreducible algebraic curve in X and let z be a point in Z . Since $\mathcal{R}(X, Y)$ is dense in $C^\infty(X, Y)$, we can find a sequence $\{\varphi_n\}$ of regular maps $\varphi_n : Z \rightarrow Y$ such that $\varphi_n(z)$ is in U_n and the composition $\varphi \circ \varphi_n$ is nonconstant. Then $\{\varphi \circ \varphi_n\}$ is an infinite family of distinct elements of $\mathcal{R}^*(Z, C)$, which is impossible in view of (i). This contradiction completes the proof. \square

Proof of Theorem 3. Let C be a compact connected affine nonsingular real algebraic curve with $g(C) \geq 2$. Let $f : M \rightarrow C$ be a nonconstant C^∞ map. Then given a neighborhood \mathcal{U} of f in $C^\infty(M, C)$, one can find an affine nonsingular real algebraic variety Y , a C^∞ diffeomorphism $\sigma : M \rightarrow Y$, and a regular map $\varphi : Y \rightarrow C$ such that $\varphi \circ \sigma$ is in \mathcal{U} (one embeds M in \mathbb{R}^n , for some n , and applies [1, Theorem 2.8.4], observing first that the bordism class of (f, M) in the unoriented bordism group of C is algebraic [1, Lemma 2.7.1]). We may assume that φ is nonconstant by taking \mathcal{U} sufficiently small. Moreover, Y must be irreducible since M is connected. We now complete the proof of Theorem 3 by applying Lemma 4. \square

We say that a closed C^∞ manifold M admits a *Weierstrass algebraic model* if there exists an affine nonsingular real algebraic variety Y such that M is diffeomorphic to Y and the set $\mathcal{R}(X, Y)$ is dense in $C^\infty(X, Y)$ for some affine nonsingular real algebraic variety X with $\dim X \geq 1$. It is likely that only very few closed C^∞ manifolds admit Weierstrass algebraic models. We conjecture that among closed connected orientable C^∞ surfaces only the 2-sphere S^2 and the torus $S^1 \times S^1$ admit Weierstrass algebraic models.

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