

SPECIAL-VALUED SUBGROUPS OF LATTICE-ORDERED GROUPS

YUANQIAN CHEN AND PAUL CONRAD

(Communicated by Lance W. Small)

ABSTRACT. We prove that the intersection of all maximal special-valued subgroups of a lattice-ordered group G is the special-valued quasi-torsion radical of a lattice-ordered group G , which extends our earlier result that the intersection of all maximal finite-valued subgroups of a lattice-ordered group G is the finite-valued torsion radical of G . We also show that the class A_f of almost finite-valued lattice-ordered groups is a quasi-torsion class, and the A_f quasi-torsion radical of a group is equal to the intersection of the group with the lateral completion of the finite-valued torsion radical of the group.

INTRODUCTION

For the basic definitions and results in lattice-ordered group theory, see M. Anderson and T. Feil [2] and M. Darnel [8]. A *lattice-ordered group*, written ℓ -group, is a partially ordered group (G, \leq) where the partial order is a lattice (meaning that each pair of elements a, b of G has a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$). An ℓ -subgroup A of an ℓ -group G is both a subgroup and a sublattice of G . A is a *convex ℓ -subgroup* of G if $a, b \in A$ and $a \leq g \leq b$ imply that $g \in A$. A normal convex ℓ -subgroup is an ℓ -ideal. A convex ℓ -subgroup which is maximal with respect to not containing some $g \in G$ is called *regular* and is a *value* of g . A regular subgroup A is an *essential value* if it contains all the values for some $g \in G$. Element g is *special* if it has a unique value and in this case the value is called a *special value*. Regular subgroups of G form a *root system* under inclusion, written $\Gamma(G)$. (That is, $\Gamma(G)$ is a partially ordered set for which $\{\alpha \in \Gamma(G) \mid \alpha \geq \gamma\}$ is totally ordered, for any $\gamma \in \Gamma(G)$.) A subset $\Delta \subseteq \Gamma(G)$ is *plenary* if $\cap \Delta = \{0\}$ and Δ is a dual ideal in $\Gamma(G)$; that is, if $\delta \in \Delta, \gamma \in \Gamma(G)$ and $\gamma > \delta$, then $\gamma \in \Delta$. If G is an abelian ℓ -group, then G is ℓ -isomorphic to an ℓ -subgroup of $V(\Gamma(G), R)$ such that if γ is a value of $g \in G$, then γ is a maximal component of g after the embedding, where $V(\Gamma(G), R)$ is the abelian ℓ -group of all real-valued functions v on $\Gamma(G)$ for which $v(\gamma) \in R$ and the support of each v satisfies the ascending chain condition. This is a consequence of the Conrad-Harvey-Holland embedding theorem for abelian lattice-ordered groups.

Received by the editors July 16, 1996 and, in revised form, September 2, 1997.

1991 *Mathematics Subject Classification*. Primary 06F15, 06F20; Secondary 20F60.

Key words and phrases. Torsion class and quasi-torsion class, finite-valued and special-valued subgroups of a lattice-ordered group.

$\Sigma(\Delta, R)$ is the ℓ -subgroup of $V(\Delta, R)$ containing all elements $v \in V$ with finite supports. $F(\Delta, R)$ is the ℓ -subgroup of $V(\Delta, R)$ containing all elements $v \in V$ whose supports are contained in a finite number of chains in Δ .

For any $g \in G$, $G(g) = \{h \in G \mid |h| \leq n |g|\}$, for some positive integer n , the *principal convex ℓ -subgroup of G generated by g* , is the least convex ℓ -subgroup of G that contains g .

An ℓ -group G is *finite-valued* if every element of G has only a finite number of values; this is equivalent to the statement that every element of G can be expressed as a finite sum of disjoint special elements. Each element of G is also called finite-valued. An ℓ -group G is *special-valued* if G has a plenary subset of special values; this is equivalent to the statement that each positive element of G can be expressed as the join of a set of pairwise disjoint positive special elements. A positive element g of G is special-valued if g can be expressed as the join of disjoint special elements.

A lattice homomorphism is *complete* if it preserves all (not necessarily finite) meets and joins. A convex ℓ -subgroup is *closed* if it is closed with respect to infinite meets and joins which exist in the ℓ -group. A convex ℓ -subgroup C of an ℓ -group G is closed if and only if the natural lattice homomorphism from G onto its lattice G/C of right cosets is complete. Extensions which preserve the lattice of closed convex ℓ -subgroups are called *a^* -extensions*.

An ℓ -group is *laterally complete* (*conditionally laterally complete*) if for any subset (bounded subset) $\{g_\alpha \mid \alpha \in A\}$ of disjoint positive elements, $\bigvee_A g_\alpha$ exists.

A *torsion class* is a class of lattice-ordered groups that is closed under convex ℓ -subgroups, ℓ -homomorphic images, and joins of convex ℓ -subgroups. For an ℓ -group G and a torsion class T , $T(G)$ indicates the join of all the convex ℓ -subgroups of G that belong to T . $T(G)$ is then the largest convex ℓ -subgroup of G that belongs to T , called the *torsion radical* of G . A *quasi-torsion class* is a class of ℓ -groups which is closed under convex ℓ -subgroups, complete ℓ -homomorphic images, and joins of convex ℓ -subgroups. For an ℓ -group G and a quasi-torsion class Q , $Q(G)$ indicates the join of all the convex ℓ -subgroups of G that belong to Q . $Q(G)$ is then the largest convex ℓ -subgroup of G that belongs to Q , called the *quasi-torsion radical* of G . Finite-valued ℓ -groups form a torsion class F_v , and special-valued ℓ -groups form a quasi-torsion class S .

We have shown that the finite-valued torsion radical of an ℓ -group G is the intersection of all maximal finite-valued subgroups of G [6]. Let S be the quasi-torsion class of special-valued ℓ -groups. We will show that the quasi-torsion radical $S(G)$ is the intersection of all the maximal special-valued subgroups of G . We will show that the class A_f of almost finite-valued ℓ -groups is a quasi-torsion class, and its quasi-torsion radical of G is equal to the intersection of G with the lateral completion of the finite-valued torsion radical of G : $A_f(G) = F_v(G)^L \cap G$. Also for each ℓ -group G , the following are equivalent:

1. There exists a largest finite-valued subgroup of G .
2. The set Δ of special values of G is an ℓ -ideal of $\Gamma(G)$.
3. $S(G)$ is the largest special-valued subgroup of G and $S(G)$ is almost finite-valued.

1. MAXIMAL SPECIAL-VALUED SUBGROUPS

Definition 1.1. A *special-valued subgroup* of an ℓ -group G is an ℓ -subgroup U such that for each $0 < g \in U$, $g = \bigvee_\lambda g_\lambda$, where the g_λ 's are disjoint and special in G .

Thus if $g \in U$, then $|g| = \vee_{\Lambda} g_{\lambda}$, where $g_{\lambda} \in G$ are disjoint and special, but we don't require that $g_{\lambda} \in U$.

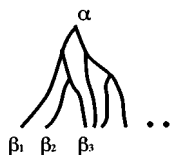
If $\dots \subseteq C_{\alpha} \subseteq C_{\beta} \subseteq C_{\gamma} \subseteq \dots$ is a chain of special-valued subgroups of G , then $\cup C_{\lambda}$ is a special-valued subgroup. Hence each special-valued subgroup is contained in a maximal special-valued subgroup.

Proposition 1.2. *If $0 < a = \vee_A a_{\alpha}$ and $0 < b = \vee_B b_{\beta}$ are special-valued elements in G , where A and B are sets of values, then $a + b$ is special-valued in G with the set of maximal elements in $A \cup B$ as special values.*

Proof. Consider

$$\begin{aligned} 0 < a + b &= \vee_A a_{\alpha} + b = \vee_A (a_{\alpha} + b) = \vee_A (a_{\alpha} + \vee_B b_{\beta}) \\ &= \vee_A (\vee_B (a_{\alpha} + b_{\beta})) = \vee_{A \cup B} (a_{\alpha} + b_{\beta}). \end{aligned}$$

Suppose that $\alpha > \beta$, for some $\beta \in B$. Then α is greater than some subset of B and disjoint from the other elements of B .



$$\begin{aligned} a_{\alpha} \wedge b &= a_{\alpha} \wedge (\vee_B b_{\beta}) = \vee_B (a_{\alpha} \wedge b_{\beta}) \\ &= (a_{\alpha} \wedge b_{\beta_1}) \vee (a_{\alpha} \wedge b_{\beta_2}) \vee (a_{\alpha} \wedge b_{\beta_3}) \vee \dots \\ &= b_{\beta_1} \vee b_{\beta_2} \vee b_{\beta_3} \vee \dots \end{aligned}$$

Thus $(a_{\alpha} + b_{\beta_1}) \vee (a_{\alpha} + b_{\beta_2}) \vee (a_{\alpha} + b_{\beta_3}) \vee \dots = a_{\alpha} + (b_{\beta_1} \vee b_{\beta_2} \vee b_{\beta_3} \vee \dots)$ is special with value α , and α is maximal in $A \cup B$.

If $\alpha = \beta$, for some $\beta \in B$, then $a_{\alpha} + b_{\beta}$ is special with value α , and α is maximal in $A \cup B$.

If α is not comparable with any β , then $a_{\alpha} + b_{\beta} = a_{\alpha} \vee b_{\beta}$ with values α and β , and both are maximal in $A \cup B$. □

Corollary 1.3. *If a and b are finite-valued, then so is $a + b$.*

Proposition 1.4. *If b is a special-valued element with $b > a > 0$, then $b + a$ is special-valued and has the same special values as b .*

Proof. Suppose that $b = \vee_{\Lambda} b_{\lambda}$ with b_{λ} disjoint and special. Then $a = b \wedge a = (\vee_{\Lambda} b_{\lambda}) \wedge a = \vee_{\Lambda} (b_{\lambda} \wedge a)$. We have

$$\begin{aligned} b + a &= b + \vee_{\Lambda} (b_{\lambda} \wedge a) \\ &= \vee_{\Lambda} (b + (b_{\lambda} \wedge a)) \\ &= \vee_{\Lambda} (\vee_{\Lambda} b_{\gamma} + (b_{\lambda} \wedge a)) \\ &= \vee_{\Lambda} (\vee_{\Lambda} (b_{\gamma} + (b_{\lambda} \wedge a))) \\ &= \vee_{\gamma \in \Lambda, \lambda \in \Lambda} (b_{\gamma} + (b_{\lambda} \wedge a)). \end{aligned}$$

If $\gamma \neq \lambda$, then $b_{\gamma} + (b_{\lambda} \wedge a) = b_{\gamma} \vee (b_{\lambda} \wedge a) \leq (b_{\gamma} + (b_{\gamma} \wedge a)) \vee (b_{\lambda} + (b_{\lambda} \wedge a))$. Therefore $b + a = \vee_{\Lambda} (b_{\lambda} + (b_{\lambda} \wedge a))$.

Now we show that $b_\lambda + (b_\lambda \wedge a)$ is special with value λ . Since $b_\lambda \in G^\lambda \setminus G_\lambda$, where G_λ is the regular subgroup of G , and G^λ the cover of G_λ , we have $b_\lambda + (b_\lambda \wedge a) \in G^\lambda \setminus G_\lambda$. Now let α be a value of $b_\lambda + (b_\lambda \wedge a)$. Then $b_\lambda \notin G_\alpha$, so $G_\alpha \subseteq G_\lambda$. If $G_\alpha \subset G_\lambda$, then $b_\lambda \leq b_\lambda + (b_\lambda \wedge a) \in G^\alpha \subseteq G_\lambda$. This contradicts the fact that $b_\lambda \in G^\lambda \setminus G_\lambda$. Therefore $\alpha = \lambda$. \square

Proposition 1.5. *An ℓ -subgroup C is the largest special-valued subgroup of G if and only if $C = S(G)$ which consists of all the special-valued elements of G .*

Proof. (\Leftarrow) If C consists of all the special-valued elements of G , then it is the largest special-valued subgroup of G .

(\Rightarrow) If $0 < g$ is special-valued, then the ℓ -subgroup $\langle g \rangle$ of G generated by g is a special-valued subgroup of G . Therefore $\langle g \rangle \subseteq C$, and hence C consists of all the special-valued elements of G .

If b is special-valued and $b > a > 0$, then by the last proposition, $b, b + a \in C$, so $a \in C$. Thus C is convex and hence $C \subseteq S(G)$. Since $S(G)$ is a special-valued subgroup of G , $S(G) \subseteq C$. \square

Corollary 1.6. *For an ℓ -group G , the following are equivalent:*

1. *There exists a largest special-valued subgroup of G .*
2. *$S(G)$ consists of all the special-valued elements of G .*
3. *$b \in G$ is special and $b > a > 0$ imply that a is special-valued.*
4. *$S(G)$ contains all the special elements of G .*

Proof. By the above proposition $1 \iff 2$, and clearly $2 \implies 3$.

$3 \implies 4$. b is special implies that $G(b) \subseteq S(G)$, so $S(G)$ contains all the special elements of G .

$4 \implies 2$. $0 < g$ is special-valued implies that $g = \vee_\Lambda g_\lambda$. Each $g_\lambda \in S(G)$, and $S(G)$ is closed. Hence, we have $g \in S(G)$. \square

Proposition 1.7. *Let G be an ℓ -group. If a is a positive special-valued element of G , b is a negative element of $S(G)$, and $a + b$ is positive, then $a + b$ is special-valued in G .*

Proof. $0 < a = \vee_A a_\alpha$, where a_α are disjoint and special with value α , and $0 < -b = \vee_B (-b_\beta)$, where $-b_\beta$ are disjoint and special with value β .

Now $0 < a + b = \vee_A a_\alpha + b$, so $0 < -b < \vee_A a_\alpha$, and hence by Proposition 1.4, $(\vee_A a_\alpha - b)$ is special-valued with the same set of special values as $\vee_A a_\alpha$. Thus A is the set of special values for $\vee_A a_\alpha - b = \vee_A a_\alpha + \vee_B (-b_\beta)$. So each β is less than or equal to one and only one α and is incomparable with the other α 's. Now consider

$$0 < a + b = \vee_A a_\alpha + b = \vee_A (a_\alpha + b) = \vee_A ((a_\alpha + b) \vee 0).$$

Case I. $a_\alpha \wedge |b| = 0$. In this case α is incomparable with all β , thus $(a_\alpha + b) \vee 0 = a_\alpha$.

Case II. $\alpha = \beta$ for some β . We then have

$$0 < -b = \vee_B (-b_\beta) = -b_\beta + \vee_{B \setminus \{\beta\}} (-b_\gamma),$$

where $-b_\beta$ and $\vee_{B \setminus \{\beta\}} (-b_\gamma)$ are disjoint. Hence

$$(a_\alpha + b) \vee 0 = (a_\alpha + b_\beta - \vee_{B \setminus \{\beta\}} (-b_\gamma)) \vee 0 = (a_\alpha + b_\beta) \vee 0$$

is a positive element in $S(G)$ with all special values less than or equal to α , since $a_\alpha + b_\beta$ is disjoint from $\vee_{B \setminus \{\beta\}} (-b_\gamma)$.

Case III. $\alpha > \beta$ for some β . Let $B_\alpha = \{\beta \in B \mid \alpha > \beta\}$. Hence for $\beta \in B \setminus B_\alpha$, $\alpha \parallel \beta$.

We now have that $-b = \vee_B(-b_\beta)$ and that $a_\alpha \wedge (-b) = a_\alpha \wedge \vee_B(-b_\beta) = \vee_B(a_\alpha \wedge (-b_\beta)) = \vee_{B_\alpha}(-b_\beta)$, so $-b = \vee_{B_\alpha}(-b_\beta) + \vee_{B \setminus B_\alpha}(-b_\beta)$ and $a_\alpha + b = a_\alpha - \vee_{B_\alpha}(-b_\beta) - \vee_{B \setminus B_\alpha}(-b_\beta)$, where $a_\alpha - \vee_{B_\alpha}(-b_\beta)$ and $\vee_{B \setminus B_\alpha}(-b_\beta)$ are positive and disjoint. Therefore, $(a_\alpha + b) \vee 0 = a_\alpha - \vee_{B_\alpha}(-b_\beta)$ is special with value α .

Thus $(a_\alpha + b) \vee 0$ can be written as the join of disjoint special elements and has all its special values less than or equal to α . Hence $0 < a + b = \vee_A((a_\alpha + b) \vee 0)$ can be written as the join of disjoint special elements, hence is special valued. \square

Proposition 1.8. *If U is a special-valued subgroup of G , then $U + S(G)$ is a special-valued subgroup of G .*

Proof. Since $S(G)$ is an ℓ -ideal, we have that $U + S(G)$ is an ℓ -subgroup of G . Consider $0 < g = a + b \in U + S(G)$, with $a \in U$ and $b \in S(G)$. We have $g + S(G) = a + S(G) = a \vee 0 + S(G)$, so without loss of generality, we may assume $a > 0$. $0 < a + b = a + b^+ - b^-$, where $a + b^+$ can be written as the join of disjoint special elements of G . So by Proposition 1.7, $a + b$ can be written as the join of disjoint special elements of G . \square

We are now ready to describe the special-valued quasi-torsion radical for an ℓ -group by its maximal special-valued subgroups.

Theorem 1.9. *$S(G)$ is the intersection of all the maximal special-valued subgroups of G .*

Proof. By Proposition 1.8, $S(G)$ is contained in the intersection of all the maximal special-valued subgroups of G . Now we pick $0 < a \in G \setminus S(G)$. We need to show there exists a maximal special-valued subgroup that does not contain a . If a is not special-valued, then it does not belong to any special-valued subgroups of G . Now suppose that $a = a_1 \vee a_2 \vee a_3 \vee \dots$, where $a_i \in G$ are disjoint and special. $G(a) \not\subseteq S(G)$, so there exists $b \in G$ such that $a > b > 0$ and b is not special-valued in G . By Proposition 1.4, $a + b$ is special-valued. Thus there exists a maximal special-valued subgroup of G that contains $a + b$ but not a . \square

Proposition 1.10. *If $0 < a$ and $0 < b$ are special-valued elements in an ℓ -group G , then so are $a \wedge b$ and $a \vee b$.*

Proof. Consider $0 < \vee_A a_\alpha$ and $0 < \vee_B b_\beta$, where the a_α 's are disjoint and a_α has special value α , and the b_β 's are disjoint and b_β has special value β .

Let $C = \{\alpha \in A \mid \text{there exists } \beta \in B \text{ comparable with } \alpha\}$. Then $C = C_1 \cup C_2$, where $C_1 = \{\alpha \in C \mid \beta \leq \alpha \text{ for some } \beta \in B\}$ and $C_2 = \{\alpha \in C \mid \alpha < \beta \text{ for some } \beta \in B\}$.

We have

$$\begin{aligned} a \vee b &= (\vee_A a_\alpha) \vee (\vee_B b_\beta) = \vee_A(a_\alpha \vee (\vee_B b_\beta)) = \vee_A(\vee_B(a_\alpha \vee b_\beta)) \\ &= \vee_{A \cup B}(a_\alpha \vee b_\beta) = \vee_{A \setminus C} a_\alpha \vee_C (a_\alpha \vee b_\beta) \vee_{B \setminus C} b_\beta \\ &= \vee_{A \setminus C} a_\alpha \vee (a_\alpha \vee (\vee_{C_1} b_\beta)) \vee (b_\beta \vee (\vee_{C_2} a_\alpha)) \vee_{B \setminus C} b_\beta. \end{aligned}$$

Therefore, $a \vee b$ is special-valued with values maximal in $A \cup B$. Similarly,

$$a \wedge b = (\vee_A a_\alpha) \wedge (\vee_B b_\beta) = \vee_A(a_\alpha \wedge (\vee_B b_\beta)) = \vee_{A \cup B}(a_\alpha \wedge b_\beta) = \vee_C(a_\alpha \wedge b_\beta).$$

Therefore, $a \wedge b$ is special-valued with values minimal in C . \square

Theorem 1.11. *A maximal special-valued subgroup U of an abelian ℓ -group G is weakly saturated; i.e., special components of elements of U belong to U .*

Proof. Suppose that $v \in U$. Then $v = g_1 \vee g_2 \vee g_3 \vee \dots$, where $g_i \in G$ are disjoint and special. Assume (by way of contradiction) that $g_1 \notin U$, and let $\langle U, g_1 \rangle$ be the ℓ -subgroup of G generated by U and g_1 . Consider $0 < f \in \langle U, g_1 \rangle$. Then

$$f = f \vee 0 = \vee_I \wedge_J ((u_{ij} \pm n_{ij}g_1) \vee 0)$$

where I and J are finite index sets, $u_{ij} \in U$ and n_{ij} are positive integers. We will first show that for fixed i and j , each $(u_{ij} \pm n_{ij}g_1) \vee 0$ is special-valued.

Consider $(u \pm ng_1) \vee 0$, $u \in U$ hence $|u| = h_1 \vee h_2 \vee h_3 \vee \dots$, where $h_i \in G$ are disjoint and special.

If the value of g_1 is different from all those of u , then clearly $(u \pm ng_1) \vee 0$, is special-valued.

If the value of g_1 is the same as a value of u , say, a value of h_1 , but the value of g_1 is the same as that of $h_1 \pm ng_1$, then we have $(u \pm ng_1) \vee 0$ is special-valued.

Now consider the case when the value of g_1 is the same as that of h_1 , and values of $h_1 \pm ng_1$ are less than that of g_1 . We then have

$$(h_1 \pm ng_1) \vee 0 \leq (u \pm nv) \vee 0 = f_1 \vee f_2 \vee f_3 \vee \dots$$

where $0 < f_i \in G$ are disjoint and special, since $(u \pm nv) \vee 0 \in U$.

Let $F = \{f_j \mid f_j \leq g_1\}$, $\vee\{f_j \mid f_j \in F\}$ exists in G because $\vee\{f_j \mid f_j \in F\} = (f_1 \vee f_2 \vee f_3 \vee \dots) \wedge g_1$. We claim that

$$(h_1 \pm ng_1) \vee 0 = \vee\{f_j \mid f_j \in F\}.$$

Clearly $(h_1 \pm ng_1) \vee 0 \leq \vee\{f_j \mid f_j \in F\}$, since $(h_1 \pm nv) \vee 0 \leq f_1 \vee f_2 \vee f_3 \vee \dots$. On the other hand, each $f_j \leq (h_1 \pm ng_1) \vee 0$ implies that $\vee\{f_j \mid f_j \in F\} \leq (h_1 \pm ng_1) \vee 0$. We have shown that $(h_1 \pm ng_1) \vee 0$ is special-valued which implies that $(u \pm ng_1) \vee 0$ is special-valued. Thus by Proposition 1.10, $\vee_I \wedge_J ((u_{ij} \pm n_{ij}g_1) \vee 0)$ is special-valued in G . \square

Corollary 1.12. *A maximal special-valued subgroup U of an abelian ℓ -group is special-valued as an ℓ -group.*

Proof. Let U be a maximal special-valued subgroup of G . For each $0 < u \in U$, $u = g_1 \vee g_2 \vee g_3 \vee \dots$, where $g_i \in G$ are disjoint and special. By Theorem 1.11, each $g_i \in U$. Also each g_i is special in U . Therefore U is special-valued as an ℓ -group. \square

2. ALMOST FINITE-VALUED ℓ -GROUPS

Definition 2.1. An ℓ -group G is *almost finite-valued* if it is special-valued and the principal convex ℓ -subgroup $G(g)$ is finite-valued for each finite-valued element $g \in G$.

We use A_f to indicate the class of almost finite-valued ℓ -groups. We first list some basic properties of A_f and brief proofs.

1. $F_v \subseteq A_f \subseteq S$, where F_v is the torsion class of finite-valued ℓ -groups, and S is the quasi-torsion class of special-valued ℓ -groups.
2. A_f is closed with respect to products.

Proof. The quasi-torsion class S of special-valued ℓ -groups is closed with respect to products, and if $0 < g$ is special in a product, then it is special in one of the factors, so $G(g)$ is finite-valued. \square

3. A_f is not closed with respect to ℓ -subgroups.

For example, for each $x \in [0, 1]$, let R_x be the real numbers. Then $\prod_{x \in [0,1]} R_x \in A_f$, but $C[0, 1] \notin S$.

4. $G \in A_f$ does not imply that $G^L \in A_f$.

For example, let $\Lambda = \bigwedge \dots$. Then $\Sigma(\Lambda, R) \in A_f$, but $\Sigma(\Lambda, R)^L = V(\Lambda, R) \notin A_f$.

5. If $G \in A_f$, then $F_v(G)$ is the ℓ -ideal generated by all the $G(g)$ with g special. So $F_v(G)$ is the largest finite-valued subgroup of G .
6. A laterally complete special-valued ℓ -group is almost finite-valued if and only if its set of special values Δ contains no copy of

$$\Lambda = \bigwedge \dots$$

Proof. Suppose Λ is contained in Δ . Then let g be a special element with value δ which is the maximal element of Δ . Since G is laterally complete, $G(g)$ contains an element with an infinite number of values. On the other hand, let g be a special element with value δ , and Δ contains no copy of Λ . Then the ideal $\bar{\delta} = \{\alpha \in \Delta \mid \alpha \leq \delta\}$ contains only a finite number of roots, so $G(g)$ is finite-valued. \square

7. For a completely distributive ℓ -group G , the following are equivalent:
 - (a) G^L is almost finite-valued.
 - (b) The plenary set Δ of all essential values of $\Gamma(G)$ contains no copy of the set Λ of (6).

Proof. By [3], $\Delta \cong$ the set of all the special elements of $\Gamma(G^L)$. \square

8. If G is a special-valued vector lattice, then without loss of generality, we assume that $\Sigma(\Delta, R) \subseteq G \subseteq V(\Delta, R)$. $V(\Delta, R) \in A_f$ if and only if Δ contains no copy of Λ as in (6). Thus each abelian a^* -extension of G is almost finite-valued if and only if Δ contains no copy of Λ .

Theorem 2.2. *For an ℓ -group G , the following are equivalent:*

1. *There exists a largest finite-valued subgroup of G .*
2. *$F_v(G)$ consists of all the finite-valued elements of G .*
3. *$0 < a < b$ and b is special imply that a is finite-valued.*
4. *b is special implies that $G(b)$ is finite-valued.*
5. *$F_v(G)$ contains all the special elements of G .*
6. *The set Δ of special values of G is an ideal of $\Gamma(G)$.*
7. *$S(G)$ is the largest special-valued subgroup of G , and $S(G)$ is almost finite-valued.*

Proof. We proved in [6] that 1, 2, and 3 are equivalent, and clearly $3 \iff 4$ and $2 \iff 5$.

Now suppose $0 < g \in G$ is special with value G_α . Then $G_\alpha \longrightarrow G_\alpha \cap G(g)$ is a one-to-one order-preserving map of the regular subgroups of G that do not contain $G(g)$ onto the regular subgroups of $G(g)$.

6 \longrightarrow 4. Δ is an ℓ -ideal of $\Gamma(G)$ implies that all the regular subgroups of $G(g)$ are special if g is special. So $G(g)$ is finite-valued.

4 \longrightarrow 7. $G(g)$ is finite-valued implies that $G(g) \subseteq S(G)$. So $S(G)$ contains all the special elements.

7 \longrightarrow 4. g is special implies $g \in S(G)$. Thus $G(g) \subseteq S(G)$, hence $G(g)$ is finite-valued.

4 \longrightarrow 6 is clear. \square

This theorem shows that if G has a largest finite-valued subgroup, then G has a largest special-valued subgroup, although the converse is not true.

For example, let $\Lambda = \bigwedge \dots$

$G = V(\Lambda, R) \in S$, but G has no largest finite-valued subgroup.

Actually, there exists a largest special-valued subgroup of G if and only if for each special $b \in G$, $G(b)$ is special-valued. In turn, this is equivalent to the assertion that for each special value $\delta \in \Delta$, the ideal $\bar{\delta} = \{\gamma \in \Gamma(G) \mid \gamma \leq \delta\}$ contains a dual ideal of special elements with zero intersection.

We now give the proof that A_f is a quasi-torsion class.

Theorem 2.3. *The almost finite-valued ℓ -groups form a quasi-torsion class A_f .*

Proof. G is almost finite-valued if and only if the set Δ of special values is a plenary subset and an ideal of $\Gamma(G)$. A special-valued ℓ -group G is almost finite-valued if and only if Δ is an ideal of $\Gamma(G)$.

1. A_f is closed with respect to convex ℓ -subgroups.

Suppose C is a convex ℓ -subgroup of an almost finite-valued ℓ -group G . Then $C \in S$, since S is a quasi-torsion class. The set C_Δ of all special values of C is an ideal and a plenary subset of G_Δ , the set of all special values of G , and G_Δ is an ideal in $\Gamma(G)$. Therefore C_Δ is an ideal in $\Gamma(C)$ and a plenary subset of $\Gamma(C)$.

2. A_f is closed with respect to complete ℓ -homomorphic images.

Let C be a closed ℓ -ideal of $G \in A_f$. Then $G/C \in S$. Suppose $C + g$ is special in G/C , $g = g_1 \vee g_2 \vee g_3 \vee \dots$ in G , with not all g_i belonging to C . If there is more than one component of g not in C , say, g_1 is one of them, then $C + g_1$ is a proper component of $C + g$, so $C + g$ is not special. So without loss of generality, g is special in G . The convex ℓ -subgroup of G/C generated by $C + g$ is isomorphic to $\frac{G(g)}{C \cap G(g)}$, and it is finite-valued, since F_v is a torsion class.

3. A_f is closed with respect to joins of convex ℓ -subgroups.

$S(G)$ is the largest convex ℓ -subgroup that belongs to S . If the largest convex ℓ -subgroup of G that is almost finite-valued exists, then it is an ℓ -ideal of $S(G)$. So without loss of generality, we assume that G is special-valued. A convex ℓ -subgroup C of G is almost finite-valued if and only if C_Δ is an ideal of $\Gamma(C)$, but $\Gamma(C)$ is an ideal of $\Gamma(G)$. So C is almost finite-valued if and only if C_Δ is an ideal in $\Gamma(G)$.

If $C_1 \subseteq C_2 \subseteq C_3 \subseteq \dots$ form a chain of convex ℓ -subgroups of G that belong to A_f , then their sets of special values $C_{1\Delta} \subseteq C_{2\Delta} \subseteq C_{3\Delta} \subseteq \dots$ form a chain of ideals in $\Gamma(G)$, so $\cup C_{i\Delta}$ is an ideal in $\Gamma(G)$, hence $\cup C_i$ is almost finite-valued.

If B and C are almost finite-valued, and $0 < g \in B + C$ is special with value δ , then $g \in B$ or $g \in C$. In fact, g is equal to the sum of positive elements from $B \cup C$, and one of them must have value δ . Without loss of generality, there exists $0 < b \in B$ with a value δ , hence $nb > g$ for some $n \in N$, so $g \in B$. Thus $B(g)$ is

finite-valued, which implies $(B + C)(g)$ is finite-valued. Therefore, $B + C$ is almost finite-valued. \square

The torsion radical $F_v(G)$ is a characteristic ℓ -ideal of G , and $F_v(G) \subseteq G \subseteq G^L$, where G^L is the lateral completion of G . Actually, $F_v(G)^L \subseteq G^L$. This follows from the following proposition.

Proposition 2.4. $C \in \mathcal{C}(G)$, the set of all convex ℓ -subgroups of G , implies that $C^L \subseteq G^L$.

Proof. $C \subseteq C''$, the polar of C in G^L , and since C'' is closed, it is laterally complete. Now G is dense in G^L , so $0 < x \in C'' \subseteq G^L$ implies $x > y > 0$, for some $y \in G \cap C''$. In particular, y is not disjoint from C^+ , so $y \wedge c > 0$, for some $c \in C$. Therefore $x > y \wedge c \in C$, so C is dense in the laterally complete ℓ -group C'' , and hence $C^L \subseteq C'' \subseteq G^L$. \square

The following theorem demonstrates the relation between the torsion radicals $A_f(G)$ and $F_v(G)$.

Theorem 2.5. $A_f(G) = F_v(G)^L \cap G$.

Proof. $F_v(G)^L$ is special-valued and has the same set Δ of special values as $F_v(G)$ by the result in [3]. Consider $0 < g \in F_v(G)^L \cap G$, $g = h_1 \vee h_2 \vee h_3 \vee \dots$, where h_i are disjoint and special in $F_v(G)^L$. Now there exists a special element $0 < g_1 \in F_v(G)$ with the same value as h_1 . Thus $ng_1 > h_1$ for some $n \in N$, so $h_1 = ng_1 \wedge g \in F_v(G)$; thus $G(h_1)$ is finite-valued.

Suppose $0 < u < g$ with $u \in G$. Then $u = u \wedge g = u \wedge (\vee h_i) = \vee(u \wedge h_i)$. Each $u \wedge h_i \in G(h_i)$ is finite-valued. Therefore u is special-valued.

If $0 < a \in G(g)$ is special, then $a < ng$, so $a < nh_i$, for some $n \in N$. Thus $G(a)$ is finite-valued. Therefore, $G(g) \subseteq A_f(G)$, so $F_v(G)^L \cap G \subseteq A_f(G)$.

Conversely, consider $0 < g \in A_f(G)$. Then $g = g_1 \vee g_2 \vee g_3 \vee \dots$, where g_i are disjoint and special, and each $G(g_i)$ is finite-valued. Therefore each $g_i \in F_v(G)$, so $g \in F_v(G)^L \cap G$. \square

Thus if G is laterally complete, then $A_f(G) = F_v(G)^L$. So $G \in A_f$ if and only if $G = F_v(G)^L$. In fact, the proof of the theorem shows that $F_v(G)(1) \cap G = A_f(G)$, where $F_v(G)(1)$ is the ℓ -group generated by the joins of disjoint elements from $F_v(G)$ [9]. Therefore $F(G)(1) \cap G = F_v(G)^L \cap G$. We now discuss almost finite-valued ℓ -groups that are laterally complete.

Proposition 2.6. Let G be a special-valued laterally complete ℓ -group. If $g \in G$ is infinite valued, then $F_v(G) + g$ is not finite-valued in $G/F_v(G)$.

Proof. If $F_v(G) + g$ is finite-valued, then without loss of generality, it is special in $G/F_v(G)$. But g can be written as $a + b$ with $a \wedge b = 0$, where a and b are both infinite-valued. Thus $F_v(G) + a$ and $F_v(G) + b$ are non-zero components of $F_v(G) + g$. This contradicts the fact that $F_v(G) + g$ is special in $G/F_v(G)$. \square


Proposition 2.7. Let G be a special-valued laterally complete ℓ -group. $G \in A_f$ if and only if $G/F_v(G)$ contains no special elements.

Proof. (\implies) If $g \in G \setminus F_v(G)$, then g is infinite-valued and hence $F_v(G) + g$ is not special.

(\Leftarrow) If $0 < g \in G$ is special, then $g \in F_v(G)$. For otherwise $F_v(G) + g$ is not special in $G/F_v(G)$. Therefore $F_v(G)$ contains all the special elements of G , and hence $G \in A_f$. \square

The next theorem follows from the above theorem and propositions.

Theorem 2.8. *For a special-valued laterally complete ℓ -group G , the following are equivalent:*

1. $G \in A_f$.
2. $G = F_v(G)^L$.
3. $F_v(G)$ contains all the special elements of G .
4. The set of special values of G contains no copy of ...

In particular, this theorem holds for the ℓ -group $V(\Delta, R)$, where Δ is any root system.

REFERENCES

1. M. Anderson, P. Bixler, and P. Conrad, *Vector lattices with no proper a -subspaces*. Archiv Math. (basel), 41 (1983), 427-433. MR **85h**:06038
2. M. Anderson, and T. Feil, *Lattice-Ordered Groups: An Introduction*. D. Reidel, 1987. MR **90b**:06001
3. P. Bixler, and M. Darnel, *Special-valued ℓ -groups*. Algebra Universalis, 22 (1986), 172-191. MR **88h**:06021
4. R. Byrd, *Lattice-ordered groups. Ph.D Dissertation*. Tulane University, 1966.
5. Y. Chen, *Torsion classes of lattice-ordered groups and special-valued subgroups*. Ph.D dissertation, University of Kansas, 1992.
6. Y. Chen, P. Conrad, and M. Darnel, *Finite-valued subgroups of lattice-ordered groups*. Czech Math Journal, 46(121) 1996. CMP 97:01
7. P. Conrad, J. Harvey, and W.C. Holland, *The Hahn embedding theorem for lattice-ordered groups*. Trans. Amer. Math. Soc., 108 (1963), 143-149. MR **27**:1519
8. M. Darnel, *Theory of Lattice-Ordered Groups*. Marcel Dekker, Inc, 1994. MR **95k**:06032
9. J. Martinez, *Torsion theory for lattice-ordered groups*. Czech. Math. J., 25 (1975), 284-299. MR **52**:10536

DEPARTMENT OF MATHEMATICAL SCIENCES, CENTRAL CONNECTICUT STATE UNIVERSITY, NEW BRITAIN, CONNECTICUT 06050

E-mail address: chen@ccsua.ctstateu.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66045