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ON THE RANGE AND THE KERNEL OF THE OPERATOR $X \mapsto AXB - X$

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Abstract. Let $L(H)$ denote the algebra of (bounded linear) operators on the separable complex Hilbert space H, and let $(\mathfrak{I}; \|\,.\,\|_{\mathfrak{I}})$ denote a norm ideal in $L(H)$. For $A, B \in L(H)$, let the derivation $\delta_{A,B}\colon L(H) \to L(H)$ be defined by $\delta_{A,B}(X) = AX - XB$, and let $\Delta_{A,B} : L(H) \to L(H)$ be defined by $\Delta_{A,B}(X) = AXB - X$. The main result of this paper is to show that if A, B are contractions, then for every operator $T \in \mathfrak{I}$ such that $ATB = T$, then $\|AXB - X + T\|_{\mathfrak{I}} \geq \|T\|_{\mathfrak{I}}$ for all $X \in \mathfrak{I}$.

1. Introduction

Recently Du Hong Ke ([2]) proved that if A, B are contractions, then for every operator S such that $ASB = S$, $A^*SB^* = S$, then

$$\|AXB - X + S\| \geq \|S\| \quad \text{for all operators } X \in L(H).$$

Duggal ([4]) proved that if A, B are contractions, then $S \in C_2$ and $ASB = S$ imply

$$\|AXB - X + S\|_2^2 = \|AXB - X\|_2^2 + \|S\|_2^2,$$

for all $X \in L(H)$, where C_2 denotes the (Hilbert) space of Hilbert-Schmidt operators on H. In this note, we shall prove the following theorem.

Theorem 1. *If A, B are contractions and $(\mathfrak{I}; \|\,.\,\|_{\mathfrak{I}})$ is a norm ideal in $L(H)$ and $T \in \mathfrak{I}$ is such that $ATB = T$, then*

$$\|AXB - X + T\|_{\mathfrak{I}} \geq \|T\|_{\mathfrak{I}} \quad \text{for all operators } X \in L(H).$$

2. Some preliminaries

Definition 2.1 ([5]). A proper two-sided ideal $\mathfrak{I}$ in $L(H)$ is said to be a norm ideal if there is a norm on $\mathfrak{I}$ satisfying the following properties:

i) $(I; \|\,.\,\|_{\mathfrak{I}})$ is a Banach space;
ii) $\|AXB\|_{\mathfrak{I}} \leq \|A\|\,\|X_{\mathfrak{I}}\|\,\|B\|$ for all $A, B \in L(H)$ and for all $X \in \mathfrak{I}$;
iii) $\|X\|_{\mathfrak{I}} = \|X\|$ for X a rank one operator.

Remark. If $(\mathfrak{I}; \|\,.\,\|_{\mathfrak{I}})$ is a norm ideal, then the norm $\|\,.\,\|_{\mathfrak{I}}$ is unitarily invariant, in the sense that $\|UAV\|_{\mathfrak{I}} = \|A\|_{\mathfrak{I}}$ for all A in $\mathfrak{I}$ and unitary U, V in $L(H)$.

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Example. Each proper ideal of $L(H)$ is contained in the ideal of compact operators. For any compact operator A, denote by $s_1(A) \geq s_2(A) \geq \cdots$ the singular values of A, i.e., the eigenvalues of $(A^*A)^{\frac{1}{2}}$.

Two special families of unitarily invariant norms satisfying conditions i), ii), and iii) of Definition 2.1 are the Schatten p-norms defined as

$$\|A\|_p = \left(\sum_{j=1}^{\infty} s_j(A)^p\right)^{\frac{1}{p}}, \qquad 1 \leq p \leq \infty,$$

where by convention

$$\|A\|_\infty = \max s_j(A) = s_1(A) = \|A\|,$$

and the Ky Fan norms defined as $\|A\|_k = \sum_{j=1}^{k} s_j(A)$, $k \geq 1$.

Theorem 2.2 ([3]). a) *Let A, $B^* \in L(H)$ be contractions with C_0 completely non-unitary parts. If $\Delta_{A,B}(X) = 0$, then $\overline{\operatorname{ran} X}$ reduces A, $\ker^\perp X$ reduces B^*, and $A|_{\overline{\operatorname{ran} X}}$ and $B^*|_{\ker^\perp X}$ are unitarily equivalent operators.*

b) *Let A and B be contractions such that $\Delta_{A,B}(X) = 0$ and X a compact operator. Then the conclusions of part* a) *hold.*

3. Main results

Theorem 3.1. *If U is a unitary operator, $(\mathfrak{I}; \|\,.\,\|_\mathfrak{I})$ is a norm ideal in $L(H)$ and $T \in \mathfrak{I}$ is such that $TU = UT$, then*

$$\|UX - XU + T\|_\mathfrak{I} \geq \|T\|_\mathfrak{I} \quad \text{for all operators } X \in \mathfrak{I}.$$

Proof. The proof is similar to J. Anderson's ([1, Theorem 1.4, p. 136]).

Corollary 3.2. *If U, V are unitary operators, $(\mathfrak{I}; \|\,.\,\|_\mathfrak{I})$ is a norm ideal in $L(H)$ and $T \in \mathfrak{I}$ is such that $UT = TV$, then*

$$\|\delta_{U,V}(X) + T\|_\mathfrak{I} \geq \|T\|_\mathfrak{I} \quad \text{for all operators } X \in \mathfrak{I}.$$

Proof. On $H \oplus H$, let

$$W = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}, \quad \widetilde{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.$$

Then $\widetilde{T} \in \mathfrak{I} \oplus \mathfrak{I}$ and $W\widetilde{X} - \widetilde{X}W + \widetilde{T} = \left(\begin{smallmatrix} 0 & UX-XV+T \\ 0 & 0 \end{smallmatrix}\right)$. Since $UT = TV$, it follows that $W\widetilde{T} = \widetilde{T}W$. By Theorem 3.1 we have $\|W\widetilde{X} - \widetilde{X}W + \widetilde{T}\|_{\mathfrak{I}\oplus\mathfrak{I}} \geq \|\widetilde{T}\|_\mathfrak{I}$ and so $\|WX - XW + T\|_\mathfrak{I} \geq \|T\|_\mathfrak{I}$.

Corollary 3.3. *If U is a unitary operator, $(\mathfrak{I}; \|\,.\,\|_\mathfrak{I})$ is a norm ideal in $L(H)$, and $T \in \mathfrak{I}$ is such that $UTU = T$. Then*

$$\|UXU - X + T\|_\mathfrak{I} \geq \|T\|_\mathfrak{I} \quad \text{for all operators } X \in \mathfrak{I}.$$

Proof. Let T be such that $UTU = T$. Then $TU = U^*T$ and so

$$\begin{aligned} \|UXU - X + T\|_\mathfrak{I} &= \|U(XU - U^*X + U^*T)\|_\mathfrak{I} \\ &= \|XU - U^*X + U^*T\|_\mathfrak{I} \\ &\geq \|U^*T\|_\mathfrak{I} \quad \text{(Corollary 3.2)} \\ &\geq \|T\|_\mathfrak{I}. \end{aligned}$$

Corollary 3.4. *If U, V are unitary operators, $(\mathfrak{I}; \|\,.\,\|_{\mathfrak{I}})$ is a norm ideal in $L(H)$, and $T \in \mathfrak{I}$ is such that $UTV = T$, then*

$$\|UXV - X + T\|_{\mathfrak{I}} \geq \|T\|_{\mathfrak{I}} \quad \textit{for all operators } X \in \mathfrak{I}.$$

Proof. On $H \oplus H$, let

$$W = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}, \quad \widetilde{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.$$

Then $\widetilde{T} \in \mathfrak{I} \oplus \mathfrak{I}$ and $W\widetilde{X}W - \widetilde{X} + \widetilde{T} = \left(\begin{smallmatrix} 0 & UXV-X+T \\ 0 & 0 \end{smallmatrix}\right)$. Since $UTV = T$, it follows that $W\widetilde{T}W = \widetilde{T}$. By Corollary 3.3 we have $\|W\widetilde{X}W - \widetilde{X} + \widetilde{T}\|_{\mathfrak{I}\oplus\mathfrak{I}} \geq \|\widetilde{T}\|_{\mathfrak{I}}$ and so

$$\|UXV - X + T\|_{\mathfrak{I}} \geq \|T\|_{\mathfrak{I}}.$$

Proof of Theorem 1. Let T be an operator such that $\Delta(T) = 0$ and $T \in \mathfrak{I}$. By Theorem 2.2b) $\overline{\operatorname{ran} X}$ reduces A, $\ker^{\perp} X$ reduces B^* and $A|_{\overline{\operatorname{ran} X}}$ and $B^*|_{\ker^{\perp} X}$ are unitary operators. Put $H_1 = H = \overline{\operatorname{ran} T} \oplus \overline{\operatorname{ran} T}^{\perp}$, $H_2 = H = \ker(T)^{\perp} \oplus \ker(T)$ so that we get decompositions of operators respectively:

$$W = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad B^* = \begin{pmatrix} B_1^* & 0 \\ 0 & B_2^* \end{pmatrix}.$$

For linear operators X, T from H_2 into H_1 we have:

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}, \qquad T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

So

$$\begin{aligned} \|\Delta_{A,B}(X) + T\|_{\mathfrak{I}} &= \left\| \begin{pmatrix} T_1 + \Delta_{A_1,B_1}(X_1) & * \\ * & * \end{pmatrix} \right\|_{\mathfrak{I}} \\ &\geq \left\| \begin{pmatrix} T_1 + \Delta_{A_1,B_1}(X_1) & 0 \\ 0 & 0 \end{pmatrix} \right\|_{\mathfrak{I}} \\ &\geq \|T_1 + \Delta_{A_1,B_1}(X_1)\|_{\mathfrak{I}}. \end{aligned}$$

Since A_1, B_1 are unitary operators, then Corollary 3.4 implies that

$$\|\Delta_{A,B}(X) + T\|_{\mathfrak{I}} \geq \|T_1\|_{\mathfrak{I}} = \|T\|_{\mathfrak{I}}.$$

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