PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 7, Pages 2135–2142
S 0002-9939(99)04755-3
Article electronically published on March 3, 1999

BIFURCATIONS OF THE HILL'S REGION IN THE THREE BODY PROBLEM

CHRISTOPHER K. MCCORD

(Communicated by Mary Rees)

ABSTRACT. In the spatial three body problem, the topology of the integral manifolds $\mathfrak{M}(c,h)$ (i.e. the level sets of energy h and angular momentum c, as well as center of mass and linear momentum) and the Hill's regions $\mathfrak{H}(c,h)$ (the projection of the integral manifold onto position coordinates) depends only on the quantity $\nu = h|c|^2$. It was established by Albouy and McCord-Meyer-Wang that, for $h < 0$ and $c \neq 0$, there are exactly eight bifurcation values for ν at which the topology of the integral manifold changes. It was also shown that for each of these values, the topology of the Hill's region changes as well. In this work, it is shown that there are no other values of ν for which the topology of the Hill's region changes. That is, a bifurcation of the Hill's region occurs if and only if a bifurcation of the integral manifold occurs.

1. Introduction

The spatial three-body problem admits the ten integrals of energy, linear momentum and angular momentum. Fixing these integrals defines an eight dimensional algebraic set called the integral manifold, $\mathfrak{M}(c,h)$, which depends on the energy level h and the magnitude c of the angular momentum vector. The seven dimensional reduced integral manifold, $\mathfrak{M}_R(c,h)$, is the quotient space $\mathfrak{M}(c,h)/SO_2$ where the SO_2 action is rotation about the angular momentum vector. The Hill's region $\mathfrak{H}(c,h)$ and the reduced Hill's region $\mathfrak{H}_R(c,h)$ are the projections onto position coordinates of $\mathfrak{M}(c,h)$ and $\mathfrak{M}_R(c,h)$ respectively.

In [3], the dependence of the topology of these spaces on the parameters c and h was investigated. It is a classical result that the topology of these sets depends only on a single bifurcation parameter, $\nu = -c^2h$. There are nine special values of this parameter, $\nu_i, i = 1, \dots, 9$. At each of the special values, the geometric restrictions imposed by the integrals change. For all but one of the values ν_i, it was shown that all four of the spaces $\mathfrak{M}(c,h)$, $\mathfrak{M}_R(c,h)$, $\mathfrak{H}(c,h)$ and $\mathfrak{H}_R(c,h)$ undergo bifurcations. The first bifurcation value is $\nu_1 = 0$, marking the passage from positive to negative energy. The next three, $\nu_2 \leq \nu_3 \leq \nu_4$, correspond to "bifurcations at infinity," in which local changes near the double collisions produce changes in the homeomorphism types of all of the spaces. Similarly, at the values

Received by the editors June 11, 1997 and, in revised form, October 14, 1997.
1991 *Mathematics Subject Classification.* Primary 70F07; Secondary 57Q60, 58F14.
The author was supported in part by grants from the National Science Foundation and the Charles Phelps Taft Memorial Fund.

$\nu_6 < \nu_7 \leq \nu_8 \leq \nu_9$, local changes near the central configurations produce changes in the homeomorphism types.

However, there was one question left unresolved by this analysis. There is a value ν_5, first detected by Simo [8] and Saari [5], [6], [7] at which the local description near a configuration called the *critical configuration* changes. In [1], it was shown that these changes do not produce changes in the homeomorphism type of $\mathfrak{M}(c,h)$ or $\mathfrak{M}_R(c,h)$. The analysis in [3] showed that the homotopy type of $\mathfrak{H}(c,h)$ and $\mathfrak{H}_R(c,h)$ does not change at ν_5, but was unable to resolve whether or not the homeomorphism type of the Hill's regions changes. The purpose of this note is to resolve that question.

Theorem 1.1. *There is no change in the homeomorphism type of the Hill's region $\mathfrak{H}(c,h)$, or of the reduced Hill's region $\mathfrak{H}_R(c,h)$, at the parameter value ν_5.*

This shows that the bifurcation information in [3] is complete:

Theorem 1.2. *For nonzero angular momentum, the integral manifold $\mathfrak{M}(c,h)$, reduced integral manifold $\mathfrak{M}_R(c,h)$, Hill's region $\mathfrak{H}(c,h)$ and reduced Hill's region $\mathfrak{H}_R(c,h)$ all share a common set of bifurcation values, corresponding to bifurcations at zero energy, bifurcations at infinity and bifurcations at the central configurations.*

The remainder of the paper is devoted to the proof of Theorem 1.1. Essentially, the proof involves reducing the problem to two consecutive applications of the h-cobordism theorem, using the energy parameter to construct the cobordisms. Some preliminaries are required before the cobordism theorem can be applied, since the Hill's regions are not manifolds, and are not compact. However, both the noncompactness and the singularities occur near the collinear configurations, while all of the changes in the geometry occur near the critical configuration c_0. We can construct homeomorphisms for each of these regions, and piece them together to produce a global homeomorphism.

2. Cobordisms

For convenience, we will fix the angular momentum vector at $\hat{k}$. This will allow us to identify ν with $-h$, and use h as a parameter. We are interested in the energy range between the bifurcations at infinity and the bifurcations at the Lagrange configuration. To be concrete, choose values h_-, h_+ such that $-\nu_6 < h_- < -\nu_5 = h_0 < h_+ < -\nu_4$, and let I denote the interval from h_- to h_+. Let

$$\hat{H} = \bigcup_{h \in I} \mathfrak{H}_R(1,h) \times \{h\}$$

and let $H^- = \mathfrak{H}_R(1,h_-)$, $H^0 = \mathfrak{H}_R(1,h_0)$, and $H^+ = \mathfrak{H}_R(1,h_+)$. Then $\hat{H}$ is a cobordism between H^- and H^+. Our goal is to prove that this cobordism is homeomorphic to the product $H^0 \times I$. This section depends crucially on the notation, constructions and results of [3, §2].

Before beginning the development of this homeomorphism, we note that such a homeomorphism of reduced Hill's regions suffices.

Lemma 2.1. *If $\mathfrak{H}_R(1,h_-) \cong \mathfrak{H}_R(1,h_+)$, then $\mathfrak{H}(1,h_-) \cong \mathfrak{H}(1,h_+)$.*

Proof. Suppose $f : \mathfrak{H}_R(1,h_-) \to \mathfrak{H}_R(1,h_+)$ is a homeomorphism. $\mathfrak{H}(1,h_+)$ is a free S^1-bundle over $\mathfrak{H}_R(1,h_+)$, and is homeomorphic to its pullback $f^*\mathfrak{H}(1,h_+)$. Thus it suffices to show $\mathfrak{H}(1,h_-)$ and $f^*\mathfrak{H}(1,h_+)$ are homeomorphic S^1-bundles over $\mathfrak{H}_R(1,h_-)$. But free S^1 bundles over $\mathfrak{H}_R(1,h_-)$ are classified by their Thom class, which is an element of $H^2(\mathfrak{H}_R(1,h_-);Z)$. If $T_\pm \in H^2(\mathfrak{H}_R(1,h_\pm);Z)$ is the Thom class of $\mathfrak{H}(1,h_\pm)$, then it is enough to show that $f^*(T_+) = T_-$.

In [3, Lemma 3.2.1], it was noted that, when the base B of an S^1 bundle $E \to B$ is simply connected with $H_2(B;Z)$ free abelian, the Thom class T_E can be computed from $\pi_1(E)$. The bundles $\mathfrak{H}(1,h_\pm) \to \mathfrak{H}_R(1,h_\pm)$ satisfy these conditions: $\mathfrak{H}_R(1,h_-)$ and $\mathfrak{H}_R(1,h_+)$ are both homotopic to a wedge of 2-spheres, and hence are simply connected with free abelian homology. Finally, from [3, Lemma 3.2.2], $\pi_1(\mathfrak{H}(1,h_+)) = \pi_1(\mathfrak{H}(1,h_-)) = 0$. Thus, in both cases, the Thom class is the generator, and the isomorphism f^* carries T_+ to T_-. □

As we will be frequently referring to different energy levels simultaneously, we need to clarify the way the notation expresses the dependence of the Hill's region on the energy level. For every h, we have

$$\mathfrak{H}_R(1,h) \xrightarrow{\omega} \mathfrak{K}_R(1,h) \xrightarrow{\psi} \mathfrak{C}(1,h).$$

The projections ω and ψ depend on the energy, and when we need to make that dependence explicit, we will write ω_h and ψ_h.

The space $\mathfrak{K}_R(1,h)$ is formed from $\mathfrak{H}_R(1,h)$ by viewing the three masses as the vertices of a triangle, and identifying similar triangles with the same orientation relative to $\hat{k}$. That is, $\mathfrak{K}_R(1,h)$ is a quotient space of $\mathfrak{H}_R(1,h)$, formed by identifying triangles which differ by a dilation. This projection is a homotopy equivalence, with the fiber over a point in $\mathfrak{K}_R(1,h)$ either a single point (for points in the boundary of $\mathfrak{K}_R(1,h)$) or a closed interval (for points in the interior of $\mathfrak{K}_R(1,h)$).

There is another formulation of $\mathfrak{H}(1,h)$ and $\mathfrak{K}(1,h)$ that we will find useful. In [3, Lemma 2.1.1], it was established that there is a function $Y(Q_1,Q_2)$ such that

$$\mathfrak{H}(1,h) = \left\{(Q_1,Q_2) \in R^6 | Y(Q_1,Q_2) \le 2(U(Q_1,Q_2)+h)\right\}.$$

Using the homogeneity of Y and the potential function U, it was further shown in [3, Lemma 2.1.2] that

$$\mathfrak{K}(1,h) = \left\{(q_1,q_2) \in R^6 |\, |q_1|^2 + |q_2|^2 = 1,\ Y(q_1,q_2) \le 2(U(q_1,q_2)\rho + h\rho^2)\right\},$$

where $\rho^2 = |Q_1|^2 + |Q_2|^2$. The boundary of $\mathfrak{K}(1,h)$ consists of the collinear configurations, and the configurations

$$\partial^0\mathfrak{K}(1,h) = \left\{(q_1,q_2) \in R^6 |\, |q_1|^2 + |q_2|^2 = 1,\ Y(q_1,q_2) = 2(U(q_1,q_2)\rho + h\rho^2)\right\}.$$

The fiber $\omega^{-1}(k)$ is a point for $k \in \partial^0\mathfrak{K}(1,h)$ and a closed interval otherwise. In particular, the projection is a homotopy equivalence.

The space $\mathfrak{C}(1,h)$ is obtained from $\mathfrak{K}_R(1,h)$ by identifying all configurations that differ from each other by a rotation. While the set $\mathfrak{C}(1,h)$ depends on h in general, for all $h \in I$ it is homeomorphic to a triangle with the vertices deleted. The edges of the triangle correspond to the collinear configurations and the deleted vertices to double collisions.

Note that, for $h \in I$, the set $\mathfrak{C}(1,h)$ does not depend on h *as a set*. However, $\mathfrak{C}(1,h)$ comes with a decomposition that *does* depend on h. For each h, $\mathfrak{C}(1,h)$ decomposes into five pieces (some of which may be empty, depending on h). These sets are categorized by the homeomorphism type of the fiber $\psi^{-1}(c)$:

Set	Fiber
L	$*$
R	$S^1 \times I$
S	S^2
T	$D^2 \sqcup D^2$
∂T	$D^2 \sqcup_{S^0} D^2$

The way in which $\mathfrak{C}(1,h)$ decomposes, and the way that decomposition depends on h, are central to our understanding of the topology of the Hill's region. This decomposition is illustrated in Figure 1. For all h, the set L is the boundary of $\mathfrak{C}(1,h)$, and corresponds to the collinear configurations. At h_-, the set T is a disk centered on the critical configuration c_0, and R makes up the remainder of $\mathfrak{C}(1,h_-)$. As h_- increases to h_0, the set T shrinks to c_0, and T correspondingly expands. At h_0, T disappears, R fills out all of the planar configurations but c_0, and c_0 becomes an element of S. As h increases from h_0 to h_+, the set S expands to a disk (still centered on c_0) and R correspondingly recedes.

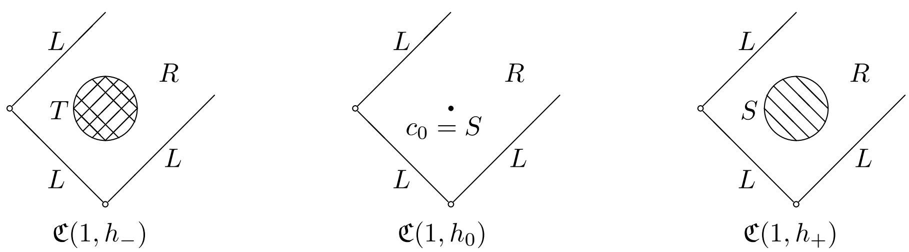

FIGURE 1. The fibers in $\mathfrak{K}_R(1,h)$ over $\mathfrak{C}(1,h)$

For any set $U \subset \mathfrak{C}$, let

$$\hat{H}(U) = \left\{(x,h) \in \hat{H} \mid \psi_h \omega_h(x) \in U\right\},$$

and let $H^-(U)$, $H^0(U)$, $H^+(U)$ denote the intersection with H^-, H^0, H^+. Two sets of this form will be of particular interest to us. Choose a disk $D \subset \mathfrak{C}$ such that

- T is contained in D for all $h_- \le h < h_0$.
- S is contained in D for all $h_0 \le h \le h_+$.

Let $C = \overline{\mathfrak{C} \setminus D}$. For convenience, we will shorten $\hat{H}(C)$ and $\hat{H}(D)$ to $\hat{C}$ and $\hat{D}$ respectively. Similarly, we will use D^-, D^0, D^+, etc. to denote the intersections with H^-, H^0, H^+. Finally, let

$$\hat{B} = \bigcup_{h \in I} \left(\partial \omega_h^{-1} \psi_h^{-1}(D)\right) \times \{h\}.$$

As the union of the boundaries of the slices of $\hat{D}$, $\hat{B}$ is itself a cobordism between $\partial \hat{H}^-(D)$ and $\partial \hat{H}^+(D)$. However, $\hat{B}$ is not quite the boundary of $\hat{D}$:

$$\partial \hat{D} = \hat{H}^-(D) \cup \hat{B} \cup \hat{H}^+(D).$$

We will construct the homeomorphism $f : \hat{H} \to H^0 \times I$ in the following stages:

1. There is a homeomorphism $f_C : \hat{C} \to C^0 \times I$ which maps $\hat{H}(C \cap D)$ to $H^0(C \cap D) \times I$.

2. This homeomorphism extends to $f_B : \hat{B} \to \big(\partial H^0(D)\big) \times I$.
3. This homeomorphism extends to $f_D : \hat{D} \to D^0 \times I$.

Because each homeomorphism extends the previous one, these all assemble to define the required homeomorphism on all of $\hat{H}$. We will construct f_C directly, then use cobordism theory to show that this can be extended first to f_B and then to f_D.

Lemma 2.2. *There is a homeomorphism $f_C : \hat{C} \to C^0 \times I$ such that $f_C\left(\hat{H}(\partial D)\right) = H^0(\partial D) \times I$.*

Proof. Since

$$\omega^{-1}\psi^{-1}(C) \cong \big(\psi^{-1}(C) \times [0,1]\big)/\partial^0\mathfrak{K}_R(C) \times [0,1],$$

it suffices to show that $\psi_h^{-1}(C) \cong \psi_{h_0}^{-1}(C)$. Our selection of C guarantees that, for every $c \in C$, the homeomorphism type of the fiber $\psi_h^{-1}(c)$ is the same for all $h_- \leq h \leq h_+$: a point if c is a collinear configuration; and an annular subset of S^2 otherwise. Moreover, the orientation of the annulus is independent of h, so the homeomorphism $\psi_h^{-1}(C) \cong \psi_{h_0}^{-1}(C)$ is constructed by simply adjusting the thickness of the annuli as needed. $\square$

The next step is to extend the homeomorphism from $\hat{H}(\partial D)$ to $\hat{B}$, and from there to $\hat{D}$. Both extensions are achieved by invoking the h-cobordism theorem. To apply the theorem, we need to know that $\hat{B}$ and $\hat{D}$ are PL manifolds with boundary. The starting point for this is the observation that the set of non-collinear configurations forms a manifold. Let

$$\mathcal{P} = \left\{(Q_1, Q_2, h) \in R^7 | 2h + 2U(Q_1, Q_2) - Y(Q_1, Q_2) \geq 0,\, Q_1 \times Q_2 \neq 0\right\}$$

and

$$\partial\mathcal{P} = \left\{(Q_1, Q_2, h) \in \mathcal{P} | 2h + 2U(Q_1, Q_2) - Y(Q_1, Q_2) = 0\right\}.$$

$\mathcal{P}$ is the non-collinear portion of the Hill's region, and $\partial\mathcal{P}$ is its boundary. Let $\mathcal{P}_R$ and $\partial\mathcal{P}_R$ denote the corresponding reduced spaces.

Proposition 2.3. *The non-collinear Hill's region $\mathcal{P}$ is a manifold with boundary $\partial\mathcal{P}$ and $\mathcal{P}_R$ is a manifold with boundary $\partial\mathcal{P}_R$.*

Proof. The Hill's region is the preimage $f^{-1}([0,\infty))$ of the function $f(Q_1, Q_2, h) = 2h + 2U(Q_1, Q_2) - Y(Q_1, Q_2)$. f is not continuous at the collinear configurations, but it is smooth on $\mathcal{P}$, which is an open set in the Hill's region. $\mathcal{P}$ is a manifold with boundary, provided f has no nonnegative critical values. But

$$\begin{aligned}&\nabla f(Q_1, Q_2, h)\\ &\quad = \left(\partial_{Q_1}\left(2U(Q_1, Q_2) - Y(Q_1, Q_2)\right), \partial_{Q_2}\left(2U(Q_1, Q_2) - Y(Q_1, Q_2)\right), 2\right)\end{aligned}$$

clearly never vanishes, so f has no critical points. Away from the collinear configurations, the SO_2 action is free, so the quotient $\mathcal{P}_R$ of the planar manifold is also a manifold. $\square$

Proposition 2.4. *$\hat{D}$, D^- and D^+ are PL-manifolds with boundary, and $\left(\hat{D}, \hat{B}\right)$ is a cobordism between $(D^-, \partial D^-)$ and $(D^+, \partial D^+)$.*

To extend $f_C : \hat{H}(\partial D) \to \hat{H}^0(\partial D) \times I$ to $f_B : \hat{B} \to \big(\partial H^0(D)\big) \times I$, we apply the 5-dimensional relative cobordism theorem [2, Proposition 7.1C]. Let

$$\hat{\beta} = cl_{\hat{B}}\hat{B} \setminus \hat{H}(\partial D) = \partial \mathcal{P}_R \cap \hat{B}$$

and let β^-, β^0, β^+ be the intersection of $\hat{\beta}$ with H^-, H^0 and H^+. To apply 5-dimensional relative cobordism theorem and conclude that $f_C|_{\hat{H}(\partial D)}$ extends to a homeomorphism f_B, it suffices to show that

- $\hat{\beta}$, β^- and β^+ simply connected.
- The inclusion maps $\beta^- \hookrightarrow \hat{\beta} \hookleftarrow \beta^+$ are homotopy equivalences.

Before establishing these results, we note that the requirements for the 6-dimensional relative cobordism theorem [4, Theorem 6.18] are similar. Once the homeomorphism f_B is shown to exist, it can be extended to a homeomorphism $f_D : \hat{D} \to D^0 \times I$ provided

- $\hat{D}$, D^- and D^+ are simply connected.
- The inclusion maps $D^- \hookrightarrow \hat{D} \hookleftarrow D^+$ are homotopy equivalences.

The proof will be complete once these results have been established.

3. Homotopy results

These calculations are all simplified by projecting the problem from $\mathfrak{H}_R(1,h)$ to $\mathfrak{K}_R(1,h)$. The projection map $\omega_h : \mathfrak{H}_R(1,h) \to \mathfrak{K}_R(1,h)$ is the identity on $\partial^0\mathfrak{K}_R$, and collapses intervals to points elsewhere. In particular, it is a homotopy equivalence, and

$$\partial \mathfrak{H}_R \cong \mathfrak{K}_R \cup_{\partial^0\mathfrak{K}_R(1,h)} \mathfrak{K}_R.$$

Let

$$\hat{\Delta} = \bigcup_{h\in I} \psi_h^{-1}(D) \times \{h\}$$

and let

$$\hat{\delta} = \bigcup_{h\in I} \big(\psi_h^{-1}(D) \cap \partial^0\mathfrak{K}_R\big) \times \{h\}.$$

Then $\hat{D} \simeq \hat{\Delta}$ and $\hat{\beta} \cong \hat{\Delta} \cup_{\hat{\delta}} \hat{\Delta}$. Similarly, there are sets Δ^-, Δ^0, Δ^+ and δ^-, δ^0, δ^+ with $D^i \simeq \Delta^i$ and $\beta^i \cong \Delta^i \cup_{\delta^i} \Delta^i$.

Lemma 3.1. *There is a strong deformation retraction $\rho : \hat{\Delta} \times [0,1] \to \Delta^0$ which restricts to a strong deformation retraction $\rho_\delta : \hat{\delta} \times [0,1] \to \delta^0$.*

Proof. First, choose a strong deformation retraction $r : D \times I \times [0,1] \to D \times \{h_0\}$ such that $r^{-1}(c_0, h) = \hat{S} \cup cl(\hat{T})$ (i.e. all configurations (c,h) such that $\psi_h^{-1}(c)$ is either a sphere or two caps), and that the restriction to $\partial D \times I$ is projection onto the first factor. We can further choose r so that rays from c_0 to ∂D are left invariant: if l is such a ray, and $\hat{l} = \bigcup_{h\in I}\{(c,h)|c \in l\}$, then each $r_t(\hat{l}) \subset \hat{l}$.

We show that r can be covered by a strong deformation retraction ρ which keeps the boundary invariant. If l is a ray from c_0 and $c \in l \cap \mathrm{int}_D(R)$, then the path $r_t(c)$ remains in $l \cap \mathrm{int}_D(R)$. Every preimage $\psi^{-1}_{r_2(c,h)}(r_1(c,h))$ is an annular subset of S^2, with fixed orientation in S^2. On such an annulus $\psi^{-1}(c,h)$, the map ρ is defined by mapping the boundary of the annulus to the boundary of the annulus $\psi^{-1}_{r_2(c,h)}(r_1(c,h))$, and extending the map linearly to the interior of the annulus.

A similar construction is applied over $\hat{S} \cup cl(\hat{T})$: over each $(c,h) \in \hat{S} \cup cl(\hat{T})$, the fiber $\psi^{-1}(c,h)$ is a subset of S^2. For $h \geq h_0$, the fiber is the entire 2-sphere, and on each 2-sphere the map ρ_t is the identity. For $h < h_0$, the fiber is either two disjoint disks (separated by the equator in S^2) or two disks touching at a pair of antipodal points in the equator. As h approaches h_0, the disks expand to fill out more and more of S^2, until they touch at all points on the equator at $\psi_{h_0}(c_0)$. On each fiber, the map ρ_t is constructed by mapping the boundary of the two disks to the boundary in $\psi^{-1}_{r_2(c,h)}(r_1(c,h))$, and extending by linearity to the interior of the disks.

Since the size and orientations of the various subsets of S^2 varies continuously with $c \in D$ and $h \in I$, all of these fiber-wise constructions are continuous, defining a single continuous strong deformation retraction ρ. $\square$

Proposition 3.2. *The sets D^-, $\hat{D}$, D^+, β^-, $\hat{\beta}$, β^+ are all simply connected.*

Proof. It suffices to show that Δ^-, Δ^0 and Δ^+ are simply connected. For D^-, $\hat{D}$, D^+, this is clearly sufficient. For β^-, $\hat{\beta}$, β^+, the decomposition $\beta^i \cong \Delta^i \cup_{\delta^i} \Delta^i$ allows us to apply the Siefert - Van Kampen theorem and conclude that $\pi_1(\beta^i)$ is a quotient of $\pi_1(\Delta^i) * \pi_1(\Delta^i)$. Clearly, if $\pi_1(\Delta^i) = 0$, then $\pi_1(\beta^i) = 0$.

As part of the proof of [3, Theorem 3.1.1], it was shown that Δ^- retracts onto $\psi^{-1}_{h_-}(\overline{T})$, and that Δ^+ retracts onto $\psi^{-1}_{h_+}(S)$. Similarly, there is a strong deformation retraction of Δ^0 onto $\psi^{-1}_{h_0}(c_0)$. For Δ^+ and Δ^0, this set is clearly homotopic to a 2-sphere (in fact, equal to a 2-sphere for Δ^0). It was also shown in the proof of [3, Theorem 3.1.1] that $\psi^{-1}_{h_-}(\overline{T})$ is homotopic to a 2-sphere. $\square$

Proposition 3.3. *The inclusion maps $D^- \hookrightarrow \hat{D} \hookleftarrow D^+$ are homotopy equivalences.*

Proof. We have reduced the calculation to showing that the maps

$$\psi^{-1}_{h_-}(\overline{T}) \xrightarrow{\rho_-} \psi^{-1}_{h_0}(c_0) \xleftarrow{\rho_+} \psi^{-1}_{h_+}(S)$$

are homotopy equivalences. The construction in [3, Theorem 3.1.1] of the 2-sphere embedded in $\psi^{-1}_{h_-}(\overline{T})$ also shows that the map ρ_- is a homotopy equivalence on that 2-sphere. When restricted to $\psi^{-1}_{h_+}(S)$, the map ρ_+ is simply a projection $S^2 \times D^2 \to S^2$, so it is also a homotopy equivalence. $\square$

Proposition 3.4. *The inclusion maps $\beta^- \hookrightarrow \hat{\beta} \hookleftarrow \beta^+$ are homotopy equivalences.*

Proof. Lemma 3.1 implies that there is a strong deformation retraction $\hat{\beta} \to \beta^0$, so it suffices to show that the maps $\beta^\pm \to \beta^0$ are homotopy equivalences. To do so, we again use the decomposition $\beta^i \cong \Delta^i \cup_{\delta^i} \Delta^i$.

The strong deformation retractions of Δ^- onto $\psi^{-1}_{h_-}(\overline{T})$, Δ^0 onto $\psi^{-1}_{h_0}(c_0)$ and Δ^+ onto $\psi^{-1}_{h_+}(S)$ in [3, Theorem 3.1.1] can also be chosen to preserve the boundary, so we can further reduce the calculations by replacing Δ^i with $\psi^{-1}(S)$ or $\psi^{-1}(\overline{T})$ and δ^i with $\psi^{-1}(S) \cap \partial^0\mathfrak{K}_R(1,h)$ or $\psi^{-1}(\overline{T}) \cap \partial^0\mathfrak{K}_R(1,h)$.

We have already noted that each Δ^i is homotopic to a 2-sphere, so it remains only to identify the sets $\psi^{-1}(S)\cap\partial^0\mathfrak{K}_R(1,h)$ and $\psi^{-1}(\overline{T})\cap\partial^0\mathfrak{K}_R(1,h)$. For both h_+ and h_0, the set $\psi^{-1}(S)\cap\partial^0\mathfrak{K}_R(1,h)$ is the equator of the 2-sphere. The set $\psi^{-1}(\overline{T})\cap \partial^0\mathfrak{K}_R(1,h_-)$ is more complicated. T is a disk, and for $t \in \overline{T}$, $\psi^{-1}(t) \cap \partial^0\mathfrak{K}_R(1,h_-)$ consists of two circles embedded in S^2, which are attached at a pair of antipodal points if and only if $t \in \partial T$. As t moves around the circle ∂T, these points precess around the equator of the 2-sphere. Thus $\psi^{-1}(\overline{T}) \cap \partial^0 \mathfrak{K}_R(1, h_-)$ can be described as $(S^1 \times D^2) \cup_{S^1} (S^1 \times D^2)$. The attaching map $S^1 \to S^1 \times D^2$ is a homotopy equivalence, so $\psi^{-1}(\overline{T}) \cap \partial^0 \mathfrak{K}_R(1, h_-)$ has a strong deformation retraction onto the equator S^1. Thus, all of the decompositions are, up to homotopy, a pair of 2-spheres attached along their equators. Since the maps preserve these decompositions, they are clearly homotopy equivalences. □

References

[1] A. Albouy, *Integral manifold of the N-body problem*, Invent. Math. **114**, (1993), 463-488. MR **95c:**58066

[2] M. H. Freedman and F. Quinn, *Topology of 4-manifolds* Princeton University Press, Princeton, NJ, 1990. MR **94b:**57021

[3] C. K. McCord, K. R. Meyer and Q. Wang, *Integral manifolds of the three body problem*. Mem. Amer. Math. Soc. **132**, (1998), 90 pp. MR **98i:**70004

[4] C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise- Linear Topology* Springer-Verlag, Berlin 1972. MR **50:**3236

[5] D. G. Saari, *From rotation and inclination to zero configurational velocity surface, II. The best possible configurational velocity surface*, Cest. Mech.**40** (1987), 197-223. MR **89j:**70016

[6] D. G. Saari, *From rotation and inclination to zero configurational velocity surface, I. A natural rotating coordinate system*, Cest. Mech. **33** (1984), 299-318. MR **86g:**70003

[7] D. G. Saari, *Symmetry in n-particle systems,* Hamiltonian Dynamical Systems: Proceedings of a Summer Research Conference held June 21-27, 1987, 23-42, edited by K.R. Meyer and D.G. Saari. MR **90b:**58087

[8] C. Simo, *El conjunto de bifurcacion en el problema espacial de tres cuerpos*, Acta I Asamblea Nacional de Astronomia y Astrofisica (1975), 211-217, Instituto de Astrofisica, Univ. de la Laguna, Spain, 211-217.

Institute for Dynamics, Department of Mathematics, University of Cincinnati, Cincinnati, Ohio 45221-0025

E-mail address: `chris.mccord@uc.edu`