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ULTRADIFFERENTIABLE FUNCTIONS ON LINES IN $\mathbb{R}^n$

TEJINDER NEELON

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ABSTRACT. It is well known that a function $f \in C^\infty(\mathbb{R}^n)$ whose restriction to every line in $\mathbb{R}^n$ is real analytic must itself be real analytic. In this note we study whether this property of real analytic functions is also possessed by some other subclasses of C^∞ functions. We prove that if $f \in C^\infty(\mathbb{R}^n)$ is ultradifferentiable corresponding to a sequence $\{M_k\}$ on every line in some 'uniform way', then f is ultradifferentiable corresponding to the sequence $\{M_k\}$.

Given an integer r, $0 \le r < \infty$, one can construct a function $u \in C^r(\mathbb{R})$ which is not real analytic but whose restriction to any line is real analytic. In 1970, Bochnak [1] and Siciak [6] independently proved that a C^∞ function that is real analytic on every line must itself be real analytic. In this note we study whether the subclass of real analytic functions can be replaced with some other subclasses of C^∞ functions. The classes we study are so-called classes of ultradifferentiable functions. Ultradifferentiable functions occur naturally in harmonic analysis, PDE, and other areas of analysis. See for example Hörmander [2], Mandelbrojt [3], Matsumoto [5], and references there. For an open set $X \subset \mathbb{R}^n$ and a sequence $\mathcal{M} = \{M_j\}_{j=0}^\infty$ of positive real numbers, define the class $C^{\mathcal{M}}(X)$ of all $u \in C^\infty(X)$ such that for every compact subset $K \subset X$ there is a constant $C_K > 0$ such that

$$\sup_{x\in K} |\partial^\alpha u(x)| \le C_K^{|\alpha|+1} M_{|\alpha|}, \forall \alpha \in \mathbb{Z}_+^n. \tag{1}$$

When $M_k = (k!)^\nu, \nu > 0$, the class $C^{\mathcal{M}}$ is called the class of Gevrey functions of order ν. When $\nu = 1$, it just the class of real analytic functions.

For $u \in C^\infty(X)$, $x \in X$, and $\xi \in \mathbb{R}^n$, define the function

$$u_{x\xi}(t) = u(x + \xi t),$$

for all $t \in \mathbb{R}$ such that $x + \xi t \in X$.

Theorem 1. *Let $X \subseteq \mathbb{R}^n$ be an open set, and let $u \in C^\infty(X)$. If $K \subset X$ is a compact subset such that for every $\xi \in \mathbb{S}^{n-1}$, there is a constant $C_{K\xi} > 0$ such that*

$$\sup_{x\in K} \left| u_{x\xi}^{(k)}(0) \right| \le C_{K\xi}^{k+1} M_k, \forall k, \tag{2}$$

then there is a constant $C_K > 0$ such that the inequality (1) *holds. Hence, if* (2) *holds for every compact subset $K \subset X$, then $u \in C^{\mathcal{M}}(X)$.*

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We will use multiindex notation: For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$, we define

$$\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}, |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n, \alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!,$$

and the binomial symbol

$$\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha_1! \alpha_2! \cdots \alpha_n!}.$$

By the Stirling formula $l! \geq e^{-l} l^l$, we have, for any $\alpha \in \mathbb{Z}_+^n$,

$$\alpha! \geq e^{-|\alpha|} \alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \cdots \alpha_n^{\alpha_n}.$$

It is easy to see that $t^t \cdot (r-t)^{r-t} \geq 2^{-r} r^r$ for all t, $r/2 \leq t \leq r$. By repeatedly applying this inequality to the above, we see that

$$\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \cdots \alpha_n^{\alpha_n} \geq 2^{-|\alpha|} |\alpha|^{|\alpha|}.$$

We get a multiindex version of Stirling formula:

$$(2e)^{-|\alpha|} |\alpha|^{|\alpha|} \leq \alpha! \leq |\alpha|^{|\alpha|}, \forall \alpha \in \mathbb{Z}_+^n. \tag{3}$$

Since there are at most $k^{n-1} \leq 2^{k(n-1)}$ multiindices of order k, (3) gives us the following:

$$\sum_{|\alpha|=k} \frac{1}{\alpha!} \leq \left(\frac{e 2^n}{k} \right)^k. \tag{4}$$

For an integer $l, 0 \leq l \leq d$, define

$$V_l(x_1, x_2, \ldots, x_d) = \det \left[x_i^j \right]_{\substack{1 \leq i \leq d \\ 0 \leq j \leq d \\ j \neq l}}.$$

When $l = d$, we have by the classical Vandermonde formula

$$V_d(x_1, x_2, \ldots, x_d) = \prod_{1 \leq i < j \leq d} (x_j - x_i). \tag{5}$$

Let $\sigma_k(x_1, x_2, \ldots, x_d) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq d} x_{i_1} x_{i_2} \cdots x_{i_k}$ denote *the* elementary symmetric polynomial of degree k in $x = (x_1, x_2, \ldots, x_d)$. The following lemma generalizes the formula (5).

Lemma 2. *$V_l(x) = V_d(x) \cdot \sigma_{d-l}(x)$ for all $l = 0, 1, \ldots, d$.*

Proof. The proof is by induction on the number of variables d. The result is obvious for $d = 1$, so assume the result is true for $d-1$ variables. Define the polynomial $P(t)$ in one variable as follows:

$$P(t) = V_l(x', t) - V_d(x', t) \cdot \sigma_{d-l}(x', t),$$

where $x' = (x_1, x_2, \ldots, x_{d-1})$. Clearly $P(x_i) = 0$ for $i = 1, 2, \ldots, d-1$. Also we claim that $P(0) = 0$. Indeed, since $V_l(x', 0) = (x_1 \cdot x_2 \cdots x_{d-1}) \cdot V_{l-1}(x')$, and $\sigma_{d-l}(x', 0) = \sigma_{d-l}(x')$, we have by induction,

$$P(0) = (x_1 \cdot x_2 \cdots x_{d-1}) \cdot V_{l-1}(x') - (x_1 \cdot x_2 \cdots x_{d-1}) \cdot V(x') \cdot \sigma_{(d-1)-(l-1)}(x') = 0.$$

Hence the polynomial $P(t)$ has d roots and is of degree at most d. To complete the proof it is enough to show that $\deg P < d$. The coefficient of t^d in $V_l(x', t)$ is $V_l(x')$, and the coefficient of t^{d-1} in $V_d(x', t)$ is $V_d(x')$. Finally the coefficient of t in $\sigma_{d-l}(x', t)$ is $\sigma_{(d-1)-l}(x')$. Hence the coefficient of t^d in $P(t)$ is equal to $V_l(x') - V(x') \cdot \sigma_{(d-1)-l}(x')$, which is zero by induction. $\square$

Lemma 3. *Let $k > 0$ be a fixed integer.*

(i) Let a, ε be real numbers with $|a| \leq 1$, $0 < \varepsilon < 1$. If $(x_0, x_1, \dots, x_k)$ is a solution of the linear system[1]

$$\sum_{j=0}^{k} \binom{k}{j} \left(a + \frac{\varepsilon i}{k}\right)^j x_j = y_i, 0 \leq i \leq k, \tag{6}$$

then

$$\max_l |x_l| \leq \left(\frac{8e^3}{\varepsilon}\right)^k \max_r |y_r|.$$

(ii) Let $a_1, a_2, ..., a_n, \varepsilon$ be real numbers with $|a_i| \leq 1, 1 \leq i \leq n, 0 < \varepsilon < 1$. For $i = 1, 2, ..., n$, set $\Gamma_i = \left\{a_{ij} = a_i + \frac{\varepsilon j}{k} : j = 0, 1, ..., k\right\}$. Any solution $\{x_\alpha\}_\alpha$ of the linear system

$$\sum_{|\alpha|=k} \binom{k}{\alpha} \xi^\alpha x_\alpha = y_\xi, \forall \xi \in \Gamma = \Gamma_1 \times \Gamma_2 \times ... \times \Gamma_n \tag{7}$$

satisfies

$$\max_{|\alpha|=k} |x_\alpha| \leq \left(\frac{8e^3}{\varepsilon}\right)^{nk} \max_{\xi \in \Gamma} \ |y_\beta|. \tag{8}$$

Proof. (i) Set $b_i = a + \frac{\varepsilon i}{k}, i = 0, 1, ..., k$. By using (5), we compute the determinant of the linear system (6):

$$\Delta = \left(\frac{\varepsilon}{k}\right)^{\frac{k(k+1)}{2}} \left(\prod_{j=0}^{k} \binom{k}{j}\right) \prod_{0 \leq i < j \leq k} (j - i).$$

Now we use Lemma 2 to evaluate Δ_{rl}, the (r, l)-th minor of Δ:

$$\Delta_{rl} = \left(\frac{\varepsilon}{k}\right)^{\frac{k(k-1)}{2}} \prod_{0 \leq j \leq k, j \neq l} \binom{k}{j} \cdot \prod_{0 \leq i < j \leq k, i, j \neq r} (j - i) \cdot \sigma_{k-l}(\widehat{b}_r),$$

where $\widehat{b}_r = (b_0, \dots, b_{(r-1)}, b_{(r+1)}, \dots, b_k)$. Since $|b_i| \leq (k + i)/k$, each term of $\sigma_{k-l}(\widehat{b}_r)$ in absolute value is at most

$$\left(\frac{1}{k}\right)^{k-l} \prod_{i=l+1}^{k} (k + i) = \left(\frac{1}{k}\right)^{k-l} \frac{(2k)!}{(k + l)!} \leq \left(\frac{1}{k}\right)^{k-l} \frac{(2k)^{2k}}{e^{-k-l}(k + l)^{k+l}} \leq \left(4e^2\right)^k.$$

Since σ_{k-l} has $\binom{k}{l}$ terms, we obtain

$$\frac{|\Delta_{rl}|}{|\Delta|} = \frac{k^k \varepsilon^{-k} \left|\sigma_{k-l}(\widehat{b}_r)\right|}{\binom{k}{l} r!(k - r)!} \leq \binom{k}{r} \left(\frac{4e^3}{\varepsilon}\right)^k. \tag{9}$$

By applying Cramer's rule we get

$$x_l = \frac{\sum_{r=0}^{k} y_r \cdot \Delta_{rl}}{\Delta}, \forall l = 0, 1, ..., k.$$

[1] Here $0^0 = 1$.

By using (9), we have for all l,

$$|x_l| \leq \max_r |y_r| \left(\frac{4e^3}{\varepsilon}\right)^k \sum_{r=0}^{k} \binom{k}{r} = \left(\frac{8e^3}{\varepsilon}\right)^k \max_r |y_r|.$$

(ii) We proceed by induction on dimension n. When $n = 1$, (7) reduces to $a_{1j}^k x_k = y_{a_{1j}}$, for all $j = 0, 1, ..., k$. By taking $j = k$, we see that (8) is trivially verified. Let $n > 1$, and assume (8) holds in dimension $n-1$. Put $\alpha' = (\alpha_2, \alpha_3, \ldots, \alpha_n)$ and $\xi' = (\xi_2, \xi_3, \ldots, \xi_n)$, and rewrite the linear system (7) as follows:

$$\sum_{\alpha_1=0}^{k} \binom{k}{\alpha_1} \xi_1^{\alpha_1} z_{_{\alpha_1 \xi'}} = y_{_{\xi_1 \xi'}}, \forall \xi_1 \in \Gamma_1, \tag{10}$$

where

(11)
$$\sum_{|\alpha'|=k-\alpha_1} \binom{k-\alpha_1}{\alpha'} (\xi')^{\alpha'} x_{\alpha_1 \alpha'} = z_{_{\alpha_1 \xi'}}, \forall \xi' \in \Gamma^{'} = \Gamma_2 \times \Gamma_3 \times ... \times \Gamma_n.$$

By applying part (i) and induction to linear systems (10) and (11), respectively, we have

$$|x_{\alpha_1 \alpha'}| \leq \left(\frac{8e^3}{\varepsilon}\right)^{(n-1)(k-\alpha_1)} \max_{\xi' \in \Gamma'} |z_{\alpha_1 \xi'}| \leq \left(\frac{8e^3}{\varepsilon}\right)^{nk} \max_{\xi \in \Gamma} \ |y_\xi|. \tag{12}$$

□

Proof of the Theorem. Since the functions

$$\xi \to u_{x\xi}^{(k)}(0) = \left([\xi_1 \partial_1 + \cdots + \xi_n \partial_n]^k u\right)(x), x \in K, k \in \mathbb{Z}_+, \tag{13}$$

are continuous, the function

$$v(\xi) = \sup_{\substack{k \in \mathbf{Z}_+ \\ x \in K}} \left(\frac{\left|u_{x\xi}^{(k)}(0)\right|}{M_k}\right)^{\frac{1}{k+1}}$$

is lower semi-continuous. In particular, the sets

$$F_m = \{\xi \in \mathbf{D}^n : v(\xi) \leq m\}, m = 1, 2, 3, ...,$$

are closed, where $\mathbf{D}^n = \{\xi \in \mathbb{R}^n : |\xi| \leq 1\}$. Observe that for $\xi \in \mathbf{D}^n$, we have $u_{x,\xi}^{(k)}(0) = |\xi|^k u_{x\frac{\xi}{|\xi|}}^{(k)}(0)$. Hence the inequality (2) also holds for all $\xi \in \mathbf{D}^n$, i.e. $\mathbf{D}^n = \bigcup_{m=1}^{\infty} F_m$. It follows from the Baire Category Theorem that there is an $m > 0$ such that the interior F_m^0 of F_m is nonempty. Let $a = (a_1, a_2, ..., a_n) \in F_m^0$, and let $\varepsilon > 0$ be such that the polydisk $[a_1, a_1 + \varepsilon] \times [a_2, a_2 + \varepsilon] \times \cdots \times [a_n, a_n + \varepsilon] \subset F_m^0$.

By (13), we have

$$u_{x\xi}^{(k)}(0) = \sum_{|\alpha|=k} \tbinom{k}{\alpha} \xi^\alpha (\partial^\alpha u)(x), \forall k, \forall \xi \in \mathbb{R}^n, x \in K. \tag{14}$$

By Lemma 3(ii), we have with $C_K = \left(\frac{8e^3}{\varepsilon}\right)^n m$,

$$\max_{\substack{|\alpha|=k \\ x\in K}} |(\partial^\alpha u)(x)| \leq \left(\frac{8e^3}{\varepsilon}\right)^{nk} \max_{\substack{\xi\in F_m \\ x\in K}} \left|u_{x\xi}^{(k)}(0)\right| \leq C_K^{k+1} M_k.$$

□

Consider the following two statements concerning the sequence $\{M_k\}$:

$$\text{(A)}\ \limsup_{k\to\infty} \frac{M_k^{\frac{1}{k}}}{k} < \infty.$$

$$\text{(S)}\ \exists R \geq 1, M_{m+n} \leq R^{m+n} M_m M_n, \forall m, n.$$

If $\{M_k\}$ satisfies (A), then $C^{\mathcal{M}}$ is a subclass of the class of real analytic functions. The condition (S), called the separativity condition (see [4]), makes the class $C^{\mathcal{M}}$ 'stable under ultradifferential' operators. The class $C^{\{1,1,1,\cdots\}}$, and Gevrey classes of order ν, $0 < \nu \leq 1$, are some of the examples of the classes whose sequences satisfy both (A) and (S). For more examples of sequences satisfying (A) and (S) see [4] and [5].

We get the following generalization of the Bochnak-Siciak theorem

Corollary 4. *Suppose $\mathcal{M} = \{M_k\}$ is a sequence satisfying both (A) and (S). If $u \in C^\infty(\mathbb{R}^n)$ is such that $u_{x_0\xi} \in C^{\mathcal{M}}(\mathbb{R})$ for every $\xi \in \mathbb{S}^{n-1}$, then u is $C^{\mathcal{M}}$ in a neighborhood of x_0.*

Proof. Let $\widetilde{u}$ denote the Taylor series expansion, around x_0, of u:

$$\widetilde{u}(x) = \sum_\alpha \frac{\partial^\alpha u(x_0)}{\alpha!} (x - x_0)^\alpha .$$

Since u satisfies the (2) with $K = \{x_0\}$, by (1) and (4) we see that $\widetilde{u}$ is majorized by the series

$$C_{x_0} \sum_{k=0}^\infty \frac{(2^n e C_{x_0})^k}{k^k} M_k \left|x - x_0\right|^k ,$$

which by (A) has a positive radius of convergence. Hence $\widetilde{u}$ represents a real analytic function near x_0. Since $C^{\mathcal{M}}(\mathbb{R})$ is a subclass of the class of real analytic functions, for any $\xi \in \mathbb{S}^{n-1}$, we have $u_{x_0\xi} \equiv \widetilde{u}_{x_0\xi}$. Hence $u \equiv \widetilde{u}$. It remains to be shown that $\widetilde{u} \in C^{\mathcal{M}}$.

By (1), and the condition (S), we have for any $\beta \in \mathbb{Z}_+^n$,

$$\left|\partial^\beta \widetilde{u}(x)\right| \leq C_{x_0}^{|\beta|+1} R^{|\beta|} M_{|\beta|} \sum_{k=0}^\infty \frac{(2^n e C_{x_0} R)^k}{k^k} M_k \left|x - x_0\right|^k .$$

As before, the last series has a positive radius of convergence, say $\rho > 0$. Hence $\widetilde{u} \in C^{\mathcal{M}}$ in $|x - x_0| < \rho$. □

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Department of Mathematics, California State University San Marcos, San Marcos, California 92096-0001
E-mail address: NEELON@MAILHOST1.CSUSM.EDU