

THE WIENER TRANSFORM ON THE BESICOVITCH SPACES

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ABSTRACT. In his fundamental research on generalized harmonic analysis, Wiener proved that the integrated Fourier transform defined by $Wf(\gamma) = \int f(t) (e^{-2\pi i\gamma t} - \chi_{[-1,1]}(t))/(-2\pi it) dt$ is an isometry from a nonlinear space of functions of bounded average quadratic power into a nonlinear space of functions of bounded quadratic variation. We consider this Wiener transform on the larger, linear, Besicovitch spaces $\mathcal{B}_{p,q}(\mathbf{R})$ defined by the norm $\|f\|_{\mathcal{B}_{p,q}} = (\int_0^\infty (\frac{1}{2T} \int_{-T}^T |f(t)|^p dt)^{q/p} \frac{dT}{T})^{1/q}$. We prove that W maps $\mathcal{B}_{p,q}(\mathbf{R})$ continuously into the homogeneous Besov space $\dot{B}_{p',q}^{1/p'}(\mathbf{R})$ for $1 < p \leq 2$ and $1 < q \leq \infty$, and is a topological isomorphism when $p = 2$.

1. INTRODUCTION

In his fundamental research on generalized harmonic analysis, Wiener [W] proved that the integrated Fourier transform, or *Wiener transform*,

$$Wf(\gamma) = \int_{-\infty}^{\infty} f(t) \frac{e^{-2\pi i\gamma t} - \chi_{[-1,1]}(t)}{-2\pi it} dt$$

satisfies the isometry relation

$$(1) \quad \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(t)|^2 dt = \lim_{\lambda \rightarrow 0} \frac{2}{\lambda} \int_{-\infty}^{\infty} |\Delta_\lambda Wf(\gamma)|^2 d\gamma,$$

where Δ_λ is the symmetric difference operator $\Delta_\lambda F(\gamma) = F(\gamma + \lambda) - F(\gamma - \lambda)$. Equation (1) is the *Wiener–Plancherel formula*.

The space of functions for which the limit of the quadratic averages on the left-hand side of (1) exists is nonlinear [HW], and therefore cannot be dealt with using the methods of ordinary functional analysis. It is therefore natural to consider the Wiener transform on larger linear spaces. The *Marcinkiewicz space* $M^p(\mathbf{R})$ is defined by the norm $\|f\|_{M^p} = \limsup_{T \rightarrow \infty} (\frac{1}{2T} \int_{-T}^T |f(t)|^p dt)^{1/p}$. Marcinkiewicz himself referred to this space as a Besicovitch space [M]. The usual Besicovitch space formed by completing the space of almost periodic functions is closely related.

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We shall deal with another collection of closely related spaces $\mathcal{B}_{p,q}(\mathbf{R})$, defined by the norm

$$\|f\|_{\mathcal{B}_{p,q}} = \begin{cases} \left(\int_0^\infty \left(\frac{1}{2T} \int_{-T}^T |f(t)|^p dt \right)^{q/p} \frac{dT}{T} \right)^{1/q}, & q < \infty, \\ \sup_{T>0} \left(\frac{1}{2T} \int_{-T}^T |f(t)|^p dt \right)^{1/p}, & q = \infty. \end{cases}$$

By extension, we refer to $\mathcal{B}_{p,q}(\mathbf{R})$ as a Besicovitch space.

Chen and Lau [CL] considered the Wiener transform on $\mathcal{B}_{2,\infty}(\mathbf{R})$, and proved that W is a topological isomorphism of $\mathcal{B}_{2,\infty}(\mathbf{R})$ onto a space V defined by the norm $\|F\|_V = \sup_{\lambda>0} \left(\frac{2}{\lambda} \int_{-\infty}^\infty |\Delta_\lambda F(\gamma)|^2 d\gamma \right)^{1/2}$. This space V coincides with the homogeneous Besov space $\dot{B}_{2,\infty}^{1/2}(\mathbf{R})$. It is our purpose in this paper to consider the Wiener transform on the Besicovitch space $\mathcal{B}_{p,q}(\mathbf{R})$ for $1 < p \leq 2, 1 < q \leq \infty$, and to connect this study to Beurling’s fundamental work on spectral synthesis [Beur].

2. NOTATION

Feichtinger [F] has observed that the Besicovitch space $\mathcal{B}_{p,q}(\mathbf{R})$ coincides (in the sense of equivalent norms) with the “dyadic” amalgam space $(L^p(\mathbf{R}^\times), \ell^q)$, where $L^p(\mathbf{R}^\times)$ is the Lebesgue space with respect to the Haar measure $dt/|t|$ on the multiplicative group \mathbf{R}^\times of nonzero real numbers. This amalgam space is defined by the norm

$$\|f\|_{(L^p(\mathbf{R}^\times), \ell^q)} = \left(\sum_{n \in \mathbf{Z}} \left(\int_{2^n \leq |t| \leq 2^{n+1}} |f(t)|^p \frac{dt}{|t|} \right)^{q/p} \right)^{1/q}.$$

As a consequence, $\mathcal{B}_{p,q}(\mathbf{R})$ is a Banach space and, for $1 \leq p, q < \infty$, its dual is $(\mathcal{B}_{p,q}(\mathbf{R}))' = \mathcal{B}_{p',q'}(\mathbf{R})$ when the duality is defined by

$$\langle f, g \rangle = \int_0^\infty \left(\frac{1}{2T} \int_{-T}^T f(t) \overline{g(t)} dt \right) \frac{dT}{T} = \frac{1}{2} \int f(t) \overline{g(t)} \frac{dt}{|t|}.$$

As usual, $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

It is a standard fact [T] that, for the range of indices $1 \leq p, q < \infty$, and $0 < s < 1$, an equivalent norm for the homogeneous Besov space $\dot{B}_{p,q}^s(\mathbf{R})$ is given by

$$\|F\|_{\dot{B}_{p,q}^s} = \left(\int_0^\infty \left(\frac{2}{\lambda^s} \left(\int_{-\infty}^\infty |\Delta_\lambda F(\gamma)|^p d\gamma \right)^{1/p} \right)^q \frac{d\lambda}{\lambda} \right)^{1/q}.$$

The usual adjustment is made when $q = \infty$.

The absolute value of the Dirichlet kernel will play an important role in our results. We use the notation

$$S(t) = \left| \frac{\sin 2\pi t}{\pi t} \right| \quad \text{and} \quad S_\lambda(t) = \lambda S(\lambda t) = \left| \frac{\sin 2\pi \lambda t}{\pi t} \right|.$$

Given an even, nonnegative function w on \mathbf{R} , we define its greatest decreasing minorant w_* and least decreasing majorant w^* by

$$w_*(t) = \inf_{0 \leq |u| \leq |t|} w(u) \quad \text{and} \quad w^*(t) = \sup_{|t| \leq |u|} w(u).$$

Note that $w_*(t) \leq w(t) \leq w^*(t)$ for all t , and that w_*, w^* are nonnegative, even functions that are decreasing on $[0, \infty)$. Also note in particular that $S^* \in L^p(\mathbf{R})$ if $p > 1$, and that $S_* = S \cdot \chi_{[-\pi/4, \pi/4]}$.

The Fourier transform is $\check{f}(\gamma) = \int f(t) e^{-2\pi i \gamma t} dt$, and the inverse Fourier transform is $\check{f}(\gamma) = \int f(t) e^{2\pi i \gamma t} dt$.

3. THE WIENER TRANSFORM ON $\mathcal{B}_{p,q}(\mathbf{R})$

We shall prove that the Wiener transform is a topological isomorphism of the Besicovitch space $\mathcal{B}_{2,q}(\mathbf{R})$ onto the homogeneous Besov space $\dot{B}_{2,q}^{1/2}(\mathbf{R})$ for $1 < q \leq \infty$, and that W maps $\mathcal{B}_{p,q}(\mathbf{R})$ continuously into $\dot{B}_{p',q}^{1/p'}(\mathbf{R})$ for $1 < p \leq 2$ and $1 < q \leq \infty$.

The following fact will be useful: if w is an even, nonnegative function on \mathbf{R} that is decreasing on $[0, \infty)$, then for each $\alpha > 1$ we have

$$(2) \quad \sum_{n \in \mathbf{Z}} \alpha^{n-1} w(\alpha^n) \leq \frac{1}{\alpha - 1} \int_0^\infty w(t) dt \leq \sum_{n \in \mathbf{Z}} \alpha^n w(\alpha^n).$$

The following lemma establishes some basic facts about the Wiener transform.

Lemma 1. *Fix $1 < p \leq 2$ and $1 \leq q \leq \infty$. If $f \in \mathcal{B}_{p,q}(\mathbf{R})$, then the following statements hold for each $\lambda > 0$.*

- (a) $f \cdot S_\lambda \in L^p(\mathbf{R})$.
- (b) $\Delta_\lambda(Wf) = \frac{1}{2}(f \cdot S_\lambda)^\wedge \in L^{p'}(\mathbf{R})$.
- (c) $\|Wf\|_{\dot{B}_{p',q}^{1/p'}} \leq \left(\int_0^\infty \left(\frac{\lambda}{2} \int_{-\infty}^\infty |f(t)|^p S(\lambda t)^p dt \right)^{q/p} \frac{d\lambda}{\lambda} \right)^{1/q}$, with equality if $p = 2$.

Proof. (a) Set $I_n = [-2^{n+1}, -2^n] \cup [2^n, 2^{n+1}]$. Then for each $\lambda > 0$,

$$\begin{aligned} \int_{-\infty}^\infty |f(t) S_\lambda(t)|^p dt &\leq \lambda^p \sum_{n \in \mathbf{Z}} \int_{I_n} |f(t)|^p S^*(\lambda t)^p \frac{2^{n+1}}{|t|} dt \\ &\leq \lambda^p \left(\sum_{n \in \mathbf{Z}} 2^{n+1} S^*(2^n \lambda)^p \right) \left(\sup_{n \in \mathbf{Z}} \int_{I_n} |f(t)|^p \frac{dt}{|t|} \right) \\ &\leq 4\lambda^p \left(\int_0^\infty S^*(\lambda t)^p dt \right) \|f\|_{(L^p(\mathbf{R}^\times), \ell^\infty)}, \end{aligned}$$

the last inequality following from (2). Since $S^* \in L^p(\mathbf{R})$ for $p > 1$ and since $\mathcal{B}_{p,q}(\mathbf{R}) = (L^p(\mathbf{R}^\times), \ell^q) \subset (L^p(\mathbf{R}^\times), \ell^\infty)$, the result follows.

- (b) Follows from a direct computation and from part (a).
- (c) By the Hausdorff–Young inequality,

$$\begin{aligned} \|Wf\|_{\dot{B}_{p',q}^{1/p'}} &= \left(\int_0^\infty \left(\frac{2}{\lambda} \int_{-\infty}^\infty |\Delta_\lambda Wf(\gamma)|^{p'} d\gamma \right)^{q/p'} \frac{d\lambda}{\lambda} \right)^{1/q} \\ &\leq \left(\int_0^\infty \left(\frac{2}{\lambda} \right)^{q/p'} \left(\int_{-\infty}^\infty |\frac{1}{2} f(t) S_\lambda(t)|^p dt \right)^{q/p} \frac{d\lambda}{\lambda} \right)^{1/q} \\ &= \left(\int_0^\infty \left(\frac{\lambda}{2} \int_{-\infty}^\infty |f(t)|^p S(\lambda t)^p dt \right)^{q/p} \frac{d\lambda}{\lambda} \right)^{1/q}, \end{aligned}$$

with equality if $p = 2$. □

The following two results estimate integrals of the form in Lemma 1(c).

Proposition 2. *Let w be even and nonnegative on \mathbf{R} , and fix $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Then*

$$\left(\int_0^\infty \left(\frac{\lambda}{2} \int_{-\infty}^\infty |f(t)|^p w(\lambda t) dt \right)^{q/p} \frac{d\lambda}{\lambda} \right)^{1/q} \leq C \|f\|_{\mathcal{B}_{p,q}},$$

with

$$C = \begin{cases} \left(\int_0^\infty w^*(t) dt \right)^{1/p}, & \text{if } p \leq q, \\ 4^{1/q} \left(\sup_{0 \leq t \leq 1} w^*(t)^{q/p} + \int_1^\infty w^*(t)^{q/p} dt \right)^{1/q}, & \text{if } q < p. \end{cases}$$

Proof. Fix $\alpha > 1$, and define $\omega_n = w^*(\alpha^n) - w^*(\alpha^{n+1})$. Then, since w^* is even and is decreasing on $[0, \infty)$,

$$(3) \quad \sum_{n \in \mathbf{Z}} \omega_n \chi_{[-\alpha^n, \alpha^n]}(t) \leq w^*(t) \leq \sum_{n \in \mathbf{Z}} \omega_n \chi_{[-\alpha^{n+1}, \alpha^{n+1}]}(t).$$

Assume now that $1 \leq p \leq q < \infty$ (the case $q = \infty$ is similar). Then, using (3), the change of variables $T = \alpha^{n+1}/\lambda$, and the triangle inequality in the Banach space $L^{q/p}(\mathbf{R}^\times)$, we compute

$$\begin{aligned} & \left(\int_0^\infty \left(\frac{\lambda}{2} \int_{-\infty}^\infty |f(t)|^p w(\lambda t) dt \right)^{q/p} \frac{d\lambda}{\lambda} \right)^{p/q} \\ & \leq \left(\int_0^\infty \left(\frac{\lambda}{2} \sum_{n \in \mathbf{Z}} \omega_n \int_{-\alpha^{n+1}/\lambda}^{\alpha^{n+1}/\lambda} |f(t)|^p dt \right)^{q/p} \frac{d\lambda}{\lambda} \right)^{p/q} \\ & = \left(\int_0^\infty \left(\sum_{n \in \mathbf{Z}} \frac{\alpha^{n+1} \omega_n}{2T} \int_{-T}^T |f(t)|^p dt \right)^{q/p} \frac{dT}{T} \right)^{p/q} \\ & \leq \left(\sum_{n \in \mathbf{Z}} \alpha^{n+1} \omega_n \right) \|f\|_{\mathcal{B}_{p,q}}^p \\ & \leq \left(\alpha \int_0^\infty w^*(t) dt \right) \|f\|_{\mathcal{B}_{p,q}}^p, \end{aligned}$$

the last inequality following by integrating (3) over $[0, \infty)$. Letting $\alpha \rightarrow 1$ therefore gives the result.

For the case $1 \leq q < p < \infty$, we use instead the triangle inequality in the metric space $L^{q/p}(\mathbf{R}^\times)$ to derive the inequality

$$\int_0^\infty \left(\sum_{n \in \mathbf{Z}} \frac{\alpha^{n+1} \omega_n}{2T} \int_{-T}^T |f(t)|^p dt \right)^{q/p} \frac{dT}{T} \leq \sum_{n \in \mathbf{Z}} (\alpha^{n+1} \omega_n)^{q/p} \|f\|_{\mathcal{B}_{p,q}}^q.$$

Then, taking $\alpha = 2^{p/q}$, we have the estimates

$$\begin{aligned} \sum_{n > 0} (\alpha^{n+1} \omega_n)^{q/p} & \leq \sum_{n > 0} 2^{n+1} w^*(2^n)^{q/p} \leq 4 \int_1^\infty w^*(t)^{q/p} dt, \\ \sum_{n \leq 0} (\alpha^{n+1} \omega_n)^{q/p} & \leq \sum_{n \leq 0} 2^{n+1} \sup_{0 \leq t \leq 1} w^*(t)^{q/p} \leq 4 \sup_{0 \leq t \leq 1} w^*(t)^{q/p}. \end{aligned}$$

□

Proposition 3. *Let w be even and nonnegative on \mathbf{R} , and fix $1 \leq p, q \leq \infty$. Then*

$$\left(\int_0^\infty \left(\frac{\lambda}{2} \int_{-\infty}^\infty |f(t)|^p w(\lambda t) dt \right)^{q/p} \frac{d\lambda}{\lambda} \right)^{1/q} \geq \left(\sup_{b \in \mathbf{R}} b w_*(b) \right)^{1/p} \|f\|_{\mathcal{B}_{p,q}}.$$

Proof. Assume that $1 \leq p, q < \infty$ (the cases $p = \infty$ or $q = \infty$ are similar). Fix $b \in \mathbf{R}$. Then, using the change of variables $\lambda = b/T$, we compute

$$\begin{aligned} \int_0^\infty \left(\frac{\lambda}{2} \int_{-\infty}^\infty |f(t)|^p w(\lambda t) dt \right)^{q/p} \frac{d\lambda}{\lambda} &\geq \int_0^\infty \left(\frac{b}{2T} \int_{-\infty}^\infty |f(t)|^p w_*(bt/T) dt \right)^{q/p} \frac{dT}{T} \\ &\geq \int_0^\infty \left(\frac{b}{2T} \int_{-T}^T |f(t)|^p w_*(b) dt \right)^{q/p} \frac{dT}{T} \\ &= (b w_*(b))^{q/p} \|f\|_{\mathcal{B}_{p,q}}^q. \end{aligned}$$

□

Theorem 4. (a) *If $1 < p \leq 2$ and $1 < q \leq \infty$, then W maps $\mathcal{B}_{p,q}(\mathbf{R})$ continuously into $\dot{B}_{p',q}^{1/p'}(\mathbf{R})$.*

(b) *W is a topological isomorphism of $\mathcal{B}_{2,q}(\mathbf{R})$ onto $\dot{B}_{2,q}^{1/2}(\mathbf{R})$ for $1 < q \leq \infty$.*

Proof. (a) In light of Lemma 1, apply Proposition 2 using $w(t) = S(t)^p$. The result follows for $p \leq q \leq \infty$ since $(S^*)^p$ is integrable, and for $1 < q \leq p$ since $(S^*)^{p \cdot q/p} = (S^*)^q$ is integrable and bounded.

(b) Since $0 < \sup(b S_*(b)^2) < \infty$, it follows from Proposition 3 that $W^{-1} : \text{Range}(W) \rightarrow \mathcal{B}_{2,q}(\mathbf{R})$ is continuous. It therefore remains only to show that W is surjective. Fix $G \in \dot{B}_{2,q}^{1/2}(\mathbf{R})$. Then $\Delta_\lambda G \in L^2(\mathbf{R})$ for almost every λ . Set $\beta_\lambda(t) = -i \sin 2\pi\lambda t$, and define $f_\lambda = (\Delta_\lambda G)^\vee / \beta_\lambda$. Then $\beta_\mu \beta_\lambda f_\lambda = (\Delta_\mu \Delta_\lambda G)^\vee = (\Delta_\lambda \Delta_\mu G)^\vee = \beta_\lambda \beta_\mu f_\mu$ and $\beta_\lambda \beta_\mu$ is nonzero a.e., so $f = f_\lambda$ is independent of λ . Define $g(t) = -2\pi i t f(t)$. Then, since $(\Delta_\lambda G)^\vee(t) = -i(\sin 2\pi\lambda t) f(t) = \frac{\lambda}{2} S(\lambda t) g(t)$, we have by the Plancherel formula and Proposition 3 that

$$\begin{aligned} \|G\|_{\dot{B}_{2,q}^{1/2}}^q &= \int_0^\infty \left(\frac{2}{\lambda} \int_{-\infty}^\infty |\Delta_\lambda G(\gamma)|^2 d\gamma \right)^{q/2} \frac{d\lambda}{\lambda} \\ &= \int_0^\infty \left(\frac{\lambda}{2} \int_{-\infty}^\infty |g(t)|^2 S(\lambda t)^2 dt \right)^{q/2} \frac{d\lambda}{\lambda} \geq C \|g\|_{\mathcal{B}_{2,q}}^q. \end{aligned}$$

Hence $g \in \mathcal{B}_{2,q}(\mathbf{R})$, and therefore $Wg \in \dot{B}_{2,q}^{1/2}(\mathbf{R})$. Finally, $\Delta_\lambda Wg = \frac{1}{2}(g \cdot S_\lambda)^\wedge = (\beta_\lambda \cdot f)^\wedge = \Delta_\lambda G$, so $Wg = G$ in $\dot{B}_{2,q}^{1/2}(\mathbf{R})$. □

4. CONNECTION TO BEURLING'S A^p, B^p SPACES

In one of his deep investigations into spectral synthesis, Beurling [Beur] introduced the spaces

$$A^p = \bigcup_{w \in \Omega} L_{w^{1-p}}^p(\mathbf{R}) \quad \text{and} \quad B^p = \bigcap_{w \in \Omega} L_w^p(\mathbf{R}).$$

Here Ω is the class of all even, positive, integrable functions on \mathbf{R} , and the weighted L^p space $L_w^p(\mathbf{R})$ is defined by the norm $\|f\|_{L_w^p} = \left(\int |f(t)|^p w(t) dt \right)^{1/p}$. A^p is a convolution algebra contained in $L^1(\mathbf{R})$. The spaces A^p and $B^{p'}$ are duals when

the duality is defined by $\langle f, g \rangle = \int f(t) \overline{g(t)} dt$. Moreover, B^p coincides with the Besicovitch space $\mathcal{B}_{p,\infty}(\mathbf{R})$, with equality of norms [Beur].

By using clever estimates on the weights w , Beurling proved that the Fourier transform is a topological isomorphism of A^2 onto a space U defined by the norm $\|F\|_U = \int_0^\infty (\frac{2}{\lambda} \int_{-\infty}^\infty |\Delta_\lambda F(\gamma)|^2 d\gamma)^{1/2} \frac{d\lambda}{\lambda}$. This space U is the homogeneous Besov space $\dot{B}_{2,1}^{1/2}(\mathbf{R})$. We will show in this section how Beurling’s isomorphism result for the Fourier transform connects to our isomorphism result for the Wiener transform.

The following lemma establishes the relationship between A^p and $\mathcal{B}_{p,1}(\mathbf{R})$.

Lemma 5. $A^p = \{tf(t) : f \in \mathcal{B}_{p,1}(\mathbf{R})\}$, with $\|f\|_{A^p} = 2 \|tf(t)\|_{\mathcal{B}_{p,1}}$.

Proof. This is a consequence of the following facts:

- (a) $(A^p)' = B^{p'}$ under the duality $\langle f, g \rangle = \int f(t) \overline{g(t)} dt$,
- (b) $(\mathcal{B}_{p,1}(\mathbf{R}))' = \mathcal{B}_{p',\infty}(\mathbf{R})$ under the duality $\langle f, g \rangle = \frac{1}{2} \int f(t) \overline{g(t)} \frac{dt}{|t|}$,
- (c) $B^{p'} = \mathcal{B}_{p',\infty}(\mathbf{R})$, with equality of norms. □

Theorem 6. W is a topological isomorphism of $\mathcal{B}_{2,1}(\mathbf{R})$ onto $\dot{B}_{2,1}^{1/2}(\mathbf{R})$.

Proof. The fact that the Fourier transform is a topological isomorphism of A^2 onto $\dot{B}_{2,1}^{1/2}(\mathbf{R})$ implies that $C_1 \|g\|_{A^2} \leq \|\hat{g}\|_{\dot{B}_{2,1}^{1/2}} \leq C_2 \|g\|_{A^2}$ for all $g \in A^2$.

Fix now any $f \in \mathcal{B}_{2,1}(\mathbf{R})$. Then, by Lemma 5, $g(t) = f(t)/t \in A^2 \subset L^1(\mathbf{R})$, with $\|g\|_{A^2} = 2 \|f\|_{\mathcal{B}_{2,1}}$. Since $f(t)/t$ is integrable, we have that

$$\Delta_\lambda Wf(\gamma) = \int_{-\infty}^\infty \frac{f(t)}{-2\pi it} (e^{-2\pi i(\gamma+\lambda)t} - e^{-2\pi i(\gamma-\lambda)t}) dt = -\frac{1}{2\pi i} \Delta_\lambda \hat{g}(\gamma).$$

Hence, $\|Wf\|_{\dot{B}_{2,1}^{1/2}} = \frac{1}{2\pi} \|\hat{g}\|_{\dot{B}_{2,1}^{1/2}}$.

As a consequence of these remarks, it follows that W maps $\mathcal{B}_{2,1}(\mathbf{R})$ continuously into $\dot{B}_{2,1}^{1/2}(\mathbf{R})$ and is invertible on its range. The same argument as in Theorem 4(b) shows that W is surjective and completes the proof. □

5. EXTENSIONS TO HIGHER DIMENSIONS

The extension of the Wiener–Plancherel formula to higher dimensions is non-trivial. Several extensions, based on differing geometries for \mathbf{R}^d , have recently been derived [BBE], [Bene], [Benk]. For example, the “rectangular” Wiener transform of [BBE] is

$$Wf(\gamma) = \int_{\mathbf{R}^d} f(t) \mathcal{E}(t, \gamma) dt, \quad \text{where } \mathcal{E}(t, \gamma) = \prod_{i=1}^d \frac{e^{-2\pi i t_j \gamma_j} - \chi_{[-1,1]}(t_j)}{-2\pi i t_j}.$$

The Wiener–Plancherel formula is then

$$\lim_{T \rightarrow \infty} \frac{1}{|R_T|} \int_{R_T} |f(t)|^2 dt = \lim_{\lambda \rightarrow 0} \frac{2^d}{\lambda_1 \cdots \lambda_d} \int_{\mathbf{R}^d} |\Delta_\lambda Wf(\gamma)|^2 d\gamma,$$

where $R_T = [-T, T]^d$, Δ_λ is the rectangular symmetric difference operator

$$\Delta_\lambda F(\gamma) = \frac{1}{2^d} \sum_{\omega \in \{-1,1\}^d} (-1)^{|\omega|} F(\gamma + \lambda\omega),$$

and the limits are taken according to a particular natural convergence criterion.

Our results all extend to the setting of [BBE]. The Besicovitch space $\mathcal{B}_{p,q}(\mathbf{R}^d)$ is defined by the norm

$$\|f\|_{\mathcal{B}_{p,q}} = \left(\int_{\mathbf{R}_+^d} \left(\frac{1}{|R_T|} \int_{R_T} |f(t)|^p dt \right)^{q/p} \frac{dT}{T_1 \cdots T_d} \right)^{1/q},$$

and a “rectangular” Besov space norm is defined by

$$\|F\|_{\dot{B}_{p,q}^s} = \left(\int_{\mathbf{R}_+^d} \left(\frac{2^d}{(\lambda_1 \cdots \lambda_d)^s} \left(\int_{\mathbf{R}^d} |\Delta_\lambda F(\gamma)|^p d\gamma \right)^{1/p} \right)^q \frac{d\lambda}{\lambda_1 \cdots \lambda_d} \right)^{1/q}.$$

Easy extensions of our results show that this Wiener transform is a topological isomorphism of $\mathcal{B}_{2,q}(\mathbf{R}^d)$ onto $\dot{B}_{2,q}^{1/2}(\mathbf{R}^d)$ for $1 < q \leq \infty$, etc. We expect that similar results should also hold true using the spherical Wiener transform of [Bene] or [Benk].

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