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TOPOLOGICAL SEQUENCE ENTROPY FOR MAPS OF THE INTERVAL

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Abstract. A result by Franzová and Smítal shows that a continuous map of the interval into itself is chaotic if and only if its topological sequence entropy relative to a suitable increasing sequence of nonnegative integers is positive. In the present paper we prove that for any increasing sequence of nonnegative integers there exists a chaotic continuous map with zero topological sequence entropy relative to this sequence.

1. Introduction

We will be concerned with the space $C(I, I)$ of continuous maps of the interval $I = [0, 1]$ into itself. Any map from $C(I, I)$ with positive topological entropy is chaotic, but the converse is not true [S]. Contrary to this topological sequence entropy is a suitable tool for characterizing chaos. In fact, Franzová and Smítal proved the following result.

Theorem 1 ([FS]). *A map $f \in C(I, I)$ is chaotic in the sense of Li and Yorke if and only if there is an increasing sequence T of nonnegative integers such that $h_T(f)$, the topological sequence entropy of f with respect to T, is positive.*

There has appeared a natural question whether there is some universal sequence which can be taken in the theorem above. Franzová and Smítal in [FS] conjectured that the sequence $\left(2^i\right)_{i=0}^{\infty}$ played the role of this universal sequence. However, the answer to the question is negative. In fact, in the last section we prove

Theorem 6. *Let T be an arbitrary increasing sequence of nonnegative integers. Then there is a chaotic map $f \in C(I, I)$ such that $h_T(f) = 0$.*

The paper is organized as follows.

In Preliminaries we present basic notation and terminology and recall the notion of topological sequence entropy. In Theorem 2 we state one known result which will be needed in the sequel. In Section 3 we recall the definition of the adding machine on the Cantor set and consider the map g obtained from the adding machine by the linear extension to the intervals contiguous to the Cantor set. Applying the technique of blowing up orbits to the map g we get a special chaotic map f (depending

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on the point q whose orbit is blown up), and we study the properties of this map with respect to topological sequence entropy (see Theorem 3 and Corollary 4). In Section 4, for any increasing sequence T of nonnegative integers we choose the point q in a special way, and in Lemma 5 we find an upper bound for the span of the map f relative to the sequence T. Using this upper bound we finally prove Theorem 6.

2. Preliminaries

Let $\mathbb{Z}$ denote the set of all integers, $\mathbb{N}$ the set of all positive integers. For $f \in C(I,I)$, f^n denotes the n-th iterate of f and $\omega_f(x)$ denotes the ω-limit set of the trajectory $(f^n x)_{n=0}^{\infty}$ of x. $\mathrm{FOrb}_f(x) = \{y \in I; \exists i,j \in \mathbb{N} \text{ with } f^i y = f^j x\}$ is called the full orbit of x and for an interval $J \subseteq I$, $\mathrm{Orb}_f(J) = \{f^n J; n = 0,1,\dots\}$ is called the orbit of J. A periodic point of f is any $x \in I$ such that $f^n x = x$ for some $n \in \mathbb{N}$, the smallest such n is called the period of x; analogously for an interval $J \subseteq I$ satisfying $f^n J = J$ for some n. The set of all periodic points of f is denoted by $\mathrm{Per}(f)$. A map f is chaotic (in the sense of Li and Yorke, see [LY]), if there is an uncountable set S such that for any $x, y \in S$, $x \neq y$, and for any $p \in \mathrm{Per}(f)$

$$\limsup_{n\to\infty} |f^n x - f^n y| > 0,$$
$$\liminf_{n\to\infty} |f^n x - f^n y| = 0,$$
$$\limsup_{n\to\infty} |f^n x - f^n p| > 0.$$

An extensive list of equivalent conditions can be found in [FŠS]. A map is called nonchaotic if it is not chaotic. If $T = (t(i))_{i=1}^{\infty}$ is an arbitrary increasing sequence of nonnegative integers define the T-trajectory of x to be the sequence $\left(f^{t(i)}x\right)_{i=1}^{\infty}$.

Definition ([G])**.** Let $f \in C(I,I)$ and T be an arbitrary sequence of nonnegative integers.

A set $E \subseteq I$ is said to be (T,f,ε,n)-separated if for any x, $y \in E$, $x \neq y$ there is an index i, $1 \le i \le n$ such that $|f^{t(i)}x - f^{t(i)}y| > \varepsilon$. Let $\mathrm{Sep}(T,f,\varepsilon,n)$ denote the largest of cardinalities of all (T,f,ε,n)-separated sets. Put $\mathrm{Sep}(T,f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Sep}(T,f,\varepsilon,n)$.

A set $F \subseteq I$ is said to be a (T,f,ε,n)-span if for any $x \in I$ there is some $y \in F$ such that $|f^{t(i)}x - f^{t(i)}y| < \varepsilon$ for $1 \le i \le n$. Let $\mathrm{Span}(T,f,\varepsilon,n)$ denote the smallest of cardinalities of all (T,f,ε,n)-spans. Put

$$\mathrm{Span}(T,f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Span}(T,f,\varepsilon,n).$$

Then $\mathrm{Sep}(T,f) = \mathrm{Span}(T,f)$ (see [G]) and we define the topological sequence entropy of f relative to T, $h_T(f)$, to be $\mathrm{Sep}(T,f)$. If $t(i) = i-1$, $i = 1,2,\dots$ then $h_T(f)$ is the topological entropy $h(f)$ of f.

Remark. Analogously we can define topological sequence entropy for continuous maps of any compact metric space X into itself. Topological sequence entropy can be viewed as the topological entropy of the nonautonomous dynamical system given on the space X by the sequence of maps $f^{t(1)}$, $f^{t(2)-t(1)}$, $f^{t(3)-t(2)}, \dots$ (see [KS]).

Theorem 2 (see [S, Theorem 3.5])**.** *Let $f \in C(I,I)$, $h(f) = 0$ and ω be an infinite ω-limit set of f. Then there is a sequence $(J(i))_{i=1}^{\infty}$ of periodic intervals of f with the following properties:*

(i) $J(i)$ *has period* 2^i,
(ii) $J(i+1) \cup f^{2^i} J(i+1) \subseteq J(i)$,
(iii) $\omega \subseteq \cup \operatorname{Orb}_f(J(i))$.

When using this theorem we will always assume the intervals $J(i)$ to be minimal.

3. Blowing up the adding machine

In the sequel we will use the concept of the adding machine (see [GH], [M], cf. also [BS]).

Define $X = \{0,1\}^{\mathbb{N}}$ to be the set of all sequences of two symbols and $X|m = \{0,1\}^{\{1,\dots,m\}}$ for $m \in \mathbb{N}$. For $\alpha \in X$ and $m \in \mathbb{N}$ we put $\alpha|m = (\alpha(1), \dots, \alpha(m)) \in X|m$. Put $\mathrm{CS}(0) = \left[0, \frac{1}{3}\right]$, $\mathrm{CS}(1) = \left[\frac{2}{3}, 1\right]$ and $\mathrm{CS}(00) = \left[0, \frac{1}{9}\right]$, $\mathrm{CS}(01) = \left[\frac{2}{9}, \frac{1}{3}\right]$, $\mathrm{CS}(10) = \left[\frac{2}{3}, \frac{7}{9}\right]$, $\mathrm{CS}(11) = \left[\frac{8}{9}, 1\right]$, etc. Then we have for the Cantor set CS

$$\mathrm{CS} = \bigcap_{m=1}^{\infty} \bigcup_{\alpha \in X|m} \mathrm{CS}(\alpha).$$

This determines in a natural way the bijection $\varkappa$ between CS and X: assign to $q \in \mathrm{CS}$ the element $\varkappa(q) \in X$ for which $q \in \mathrm{CS}(\varkappa(q)|m)$ for every $m \in \mathbb{N}$. We call $\varkappa(q)$ the code of q and we also write $\varkappa(q) = (q(i))_{i=1}^{\infty}$. Now we can define the adding machine Add: $\mathrm{CS} \to \mathrm{CS}$ by

$$\mathrm{Add}(q) = \varkappa^{-1}(\varkappa(q) + 1),$$

where the addition is modulo 2 from the left to the right. The map Add is a bijection.

Completing the adding machine linearly on each interval contiguous to CS we get the continuous function $g : I \to I$ which then satisfies the following properties (see [BS, Section 4]):

(g1) CS is the unique infinite ω-limit set of g,
(g2) g is nonchaotic (and so $h(g) = 0$),
(g3) $\forall x \in I$: $\operatorname{Card} g^{-1}(x) \le 3$.

Now fix an arbitrary $q \in \mathrm{CS}$ such that both symbols 0 and 1 occur in its code $\varkappa(q)$ infinitely many times. Due to (g3) we can use the technique of blowing up orbits (see [H], cf. also [BS]).

$\mathrm{FOrb}_g(q)$ is countable and in the following we take its enumeration in the form $\mathrm{FOrb}_g(q) = \{q_k; k \in K\}$ where $q_i = \mathrm{Add}^i(q)$ for every $i \in \mathbb{Z}$ and $K \supseteq \mathbb{Z}$ is a suitable countable index set. Let $\{Q_k; k \in K\}$ be a system of compact pairwise disjoint subintervals of I with similar ordering as the set $\{q_k; k \in K\}$ (i.e., $Q_k < Q_l$ iff $q_k < q_l$). Now replace every q_k by Q_k and compress the rest of the interval I in such a way that we again receive I. Formally, apply to I the blowing up morphism φ, which is an increasing set-valued function assigning to any q_k the interval Q_k and to any $x \in I \setminus \mathrm{FOrb}_g(q)$ a point such that $\varphi(I) = I$ and φ is linear on every interval contiguous to $\mathrm{FOrb}_g(q)$. The map φ has an inverse φ^{-1} which is a continuous real-valued nondecreasing function in $C(I,I)$, constant on every Q_k.

Define $f \in C(I,I)$ by

$$f(x) = \begin{cases} \varphi(g(\varphi^{-1}(x))) & \text{for } x \in I \setminus \bigcup_{k\in K} Q_k, \\ \text{linearly extended} & \text{on } \bigcup_{k\in K} Q_k. \end{cases}$$

Then f has the following properties (see [BS, Theorem 6.2]):

(f1) f has the unique infinite ω-limit set $\omega = \varphi(\mathrm{CS}) \setminus \bigcup_{k\in K} \mathrm{Int}\, Q_k$,

(f2) $\omega \subsetneq \bigcap_{i=1}^{\infty} \cup \mathrm{Orb}\, J(i)$

(f3) $h(f) = 0$.

Associate with q the increasing sequence $A(q) = (a(i))_{i=1}^{\infty}$ of all indices in which the code of q "changes" the symbols, i.e., $q(a(i)+1) = 1 - q(a(i))$. Using this sequence, define another one $B(q) = (b(i))_{i=1}^{\infty}$ by putting $b(i) = 2^{a(1)} + \cdots + 2^{a(i)}$ for every $i \in \mathbb{N}$.

Theorem 3. *For f and B defined above we have $h_B(f) > 0$.*

Proof. Let $(J(i))_{i=1}^{\infty}$ be the sequence of periodic intervals of f from Theorem 2 applied to f and ω, such that $Q_0 \subseteq J(i)$ for every $i \in \mathbb{N}$. In the interval $J(i)$ there is the unique fixed point of f^{2^i}, denote it by $\mathrm{fix}(i)$. Let $\mathrm{fix}(0)$ be the unique fixed point of f. Define $L(i)$ ($R(i)$, respectively) to be the left (right, respectively) component of $\mathrm{Conv}\,(J(a(i)), \mathrm{fix}(a(i)-1)) \setminus \mathrm{Int}\, Q_0$ for every $i \in \mathbb{N}$ (here $\mathrm{Conv}\, A$ denotes the convex hull of A).

First we prove that

$$f^{2^{a(i)}} L(i) \supseteq L(i+1) \cup R(i+1), \tag{1}$$

$$f^{2^{a(i)}} R(i) \supseteq L(i+1) \cup R(i+1). \tag{2}$$

To this end we first realize that

$$\begin{aligned} \mathrm{Conv}(J(a(i)+1), \mathrm{fix}(a(i))) &\supseteq \mathrm{Conv}(J(a(i+1)), \mathrm{fix}(a(i+1)-1)) \\ &\supseteq L(i+1) \cup R(i+1). \end{aligned} \tag{3}$$

Without loss of generality we can assume that $q(a(i)) = 0$. Then $\mathrm{fix}(a(i)) \in L(i)$ and $f^{2^{a(i)}} J(a(i)+1) \subseteq L(i)$. To prove (1) it suffices to use $f^{2^{a(i)}} \mathrm{fix}(a(i)) = \mathrm{fix}(a(i))$, $f^{2^{a(i)}} \left(f^{2^{a(i)}} J(a(i)+1) \right) = J(a(i)+1)$ and (3).

From $J(a(i)+1) \cap R(i) \neq \emptyset$ we have $f^{2^{a(i)}} J(a(i)+1) \cap f^{2^{a(i)}} R(i) \neq \emptyset$. To prove (2) it is sufficient to realize that $f^{2^{a(i)}} \mathrm{fix}(a(i)-1) = \mathrm{fix}(a(i)-1) \in R(i)$, $f^{2^{a(i)}} J(a(i)+1)$ lies to the left from $\mathrm{fix}(a(i))$ and to use (3).

Now fix $n \in \mathbb{N}$ and $0 < \varepsilon < \mathrm{diam}\, Q_0$. We are going to show that $\mathrm{Sep}(B, f, \varepsilon, n) \geq 2^n$. To see this, take any sequence $(S(1), \ldots, S(n))$ where $S(i)$ is either $L(i+1)$ or $R(i+1)$, $i = 1, \ldots, n$. By (1) and (2) there is a point $x \in I$ such that $f^{b(i)} x \in S(i)$ for $i = 1, \ldots, n$. There are 2^n such sequences and so we have 2^n different points forming an (B, f, ε, n)-separated set (note that $\mathrm{dist}(L(i), R(i)) > \varepsilon$ for any i). Hence $\mathrm{Sep}(B, f, \varepsilon, n) \geq 2^n$. Since $0 < \varepsilon < \mathrm{diam}\, Q_0$ and $n \in \mathbb{N}$ were arbitrary, we get $h_B(f) \geq \log 2$. □

Corollary 4. *The map f defined above is chaotic.*

Remark. If we take q with $\varkappa(q)$ almost stationary we obtain a nonchaotic map which has the unique infinite ω-limit set, and satisfies (f2) and (f3) but not (f1).

4. The main result

Let $T = (t(i))_{i=1}^{\infty}$ be an arbitrary increasing sequence of nonnegative integers. Take any point $q \in \mathrm{CS}$ such that for the sequence $A(q) = (a(i))_{i=1}^{\infty}$ (defined in the previous section) it holds that

$$2^{a(i)} > t(2^i) \quad \text{for} \quad i = 1, 2, \ldots. \tag{4}$$

Let f be the function in $C(I,I)$ obtained from g by blowing up the full orbit $\mathrm{FOrb}_g(q)$ as in the previous section.

If $m \in \mathbb{N}$ and $\mathcal{S} = \{S_1, \ldots, S_s\}$ is a nonempty finite family of pairwise disjoint nonempty subsets of I then $\mathrm{Code}(m, \mathcal{S})$ will denote the set of all sequences $(A_1, \ldots, A_m) \in \mathcal{S}^m$ with the property that there exists a point $x \in I$ such that $f^{t(i)}x \in A_i,\ i = 1, \ldots, m$. In such a case we will say that $(A_1, \ldots, A_m)$ is the code of x (or, more precisely, the code of x of length m with symbols from $\mathcal{S}$).

Lemma 5. *For any $\varepsilon > 0$ there exists a periodic interval $J(a(p))$ of f such that* $\mathrm{Span}(T, f|\cup \mathrm{Orb}_f\, J(a(p)), \varepsilon, 2^n) = O\left(2^{n^2}\right)$.

Proof. In the sequel $J_j(i)$, where $-2^{i-1}+1 \le j \le 2^{i-1}$, will denote that interval in $\mathrm{Orb}_f\, J(i)$ which contains Q_j. Let ε be an arbitrary positive number. There is a $p \in \mathbb{N}$ such that the periodic interval $J(a(p))$ satisfies

$$\operatorname{diam} J_j(a(p)) - \operatorname{diam} Q_j < \varepsilon \quad \text{for} \quad j = -2^{a(p)-1}+1, \ldots, -1, 0, 1, \ldots, 2^{a(p)-1}.$$

To see this, suppose this is not the case. Then there is a nested sequence $(A(n))_{n=0}^{\infty}$ of periodic intervals $A(n) \in \mathrm{Orb}\, J(n)$ such that $\bigcap_{n=1}^{\infty} A(n)$ is an interval with diameter larger than the diameter of that of the intervals $Q_k,\ k \in K$, which is contained in it. But then the map g has a nested sequence $(\varphi^{-1}A(n))_{n=1}^{\infty}$ of periodic intervals whose intersection is not a point which is impossible.

Let n be an arbitrary positive integer. Put

$$M_j(i) = J_j(a(p+i-1)) \setminus J_j(a(p+i)) \quad \text{for} \quad i = 1, \ldots, n,$$
$$j = -2^{a(p)-1}+1, \ldots, -1, 0, 1, \ldots, 2^{a(p)-1}.$$

It is easy to see that the defined intervals have the following property:

(5) any periodic interval from $\mathrm{Orb}_f\, J(a(p+r)),\ 1 \le r \le n$ either is one of the intervals $J_{-2^{a(p)-1}+1}(a(p+r)), \ldots, J_{2^{a(p)-1}}(a(p+r))$, or is contained in some $M_j(s),\ 1 \le s \le r$.

Put $\mathcal{J} = \{J_{-2^{a(p)-1}+1}(a(p+n)), \ldots, J_{2^{a(p)-1}}(a(p+n))\}$, $\mathcal{M} = \bigcup_j \{M_j(1), \ldots, M_j(n)\}$ and consider the decomposition $\mathcal{D} = \mathcal{M} \cup \mathcal{J}$ of the set $\cup \mathrm{Orb}_f\, J(a(p))$. To each $x \in \cup \mathrm{Orb}_f\, J(a(p))$ assign its code $(c(1), \ldots, c(2^n)) \in \mathrm{Code}(2^n, \mathcal{D})$. From (4) it follows that any such code can contain at most $2^{a(p)}$ symbols from $\mathcal{J}$. Suppose that a code c contains a symbol from $\mathcal{J}$ and let k is the first number such that $c(k)$ belongs to $\mathcal{J}$. Then the rest of the code c, i.e. $(c(k+1), \ldots, c(2^n))$ is uniquely determined. This straightforwardly follows from (5). Thus

$$\operatorname{Card} \mathrm{Code}(2^n, \mathcal{D}) \le \sum_{m=0}^{2^n} \operatorname{Card} \mathrm{Code}(m, \mathcal{M}) \cdot \operatorname{Card} \mathcal{J}. \tag{6}$$

We claim that if $c = (M_{k_1}(c_1), \ldots, M_{k_m}(c_m)) \in \mathrm{Code}(m, \mathcal{M})$ and $c_i > c_j$ for some $1 \le i < j \le m$ then $M_{k_j}(c_j)$ is uniquely determined by $(M_{k_1}(c_1), \ldots, M_{k_i}(c_i))$.

In fact, the existence of a point with code c shows that $f^{t(i)-t(j)}M_{k_i}(c_i)\cap M_{k_j}(c_j)\neq\emptyset$. Further, $M_{k_i}(c_i)\subseteq J_{k_i}(a(p+c_i-1))$. Using $c_i-1\geq c_j$, (5) then implies $f^{t(j)-t(i)}M_{k_i}(c_i)\subseteq M_{k_j}(c_j)$.

Note that the indices $k_2,\dots,k_m$ are uniquely determined by k_1. Now fix an index k_1 and to each code $c=(M_{k_1}(c_1),\dots,M_{k_m}(c_m))\in\operatorname{Code}(m,\mathcal{M})$ assign a nondecreasing sequence $\operatorname{ris}(c)$, the rising sun sequence of c, defined by $\operatorname{ris}(c)(i)=\max\{c(1),\dots,c(i)\}$, $i=1,\dots,m$. By what we showed above, if $(M_{k_1}(c_1),\dots,M_{k_{i-1}}(c_{i-1}),M_{k_i}(c_i))$ and $(M_{k_1}(c_1),\dots,M_{k_{i-1}}(c_{i-1}),M_{k_i}(d_i))$ are two codes with $c_i\neq d_i$ then $c_i,d_i\geq c_1,\dots,c_{i-1}$. This implies that $\operatorname{ris}(c)\neq\operatorname{ris}(d)$ whenever c,d are two different codes from $\operatorname{Code}(m,\mathcal{M})$ with the same index k_1. When we realize that there are $2^{a(p)}$ possible values for k_1 we can bound above $\operatorname{Card}\operatorname{Code}(m,\mathcal{M})$ by the number of all nondecreasing sequences $\{1,\dots,m\}\to\{1,\dots,n\}$, which equals $\binom{m+n-1}{m}\leq\binom{2^n+n-1}{2^n}$, multiplied by $2^{a(p)}$. Hence, by (6)

$$\begin{aligned}\operatorname{Card}\operatorname{Code}(2^n,\mathcal{D})&\leq\sum_{m=0}^{2^n}\binom{m+n-1}{m}\cdot 2^{a(p)}\cdot 2^{a(p)}\\&\leq(2^n+1)\cdot\binom{2^n+n-1}{2^n}\cdot 2^{2\cdot a(p)}=O\left(2^{n^2}\right).\end{aligned}$$

Finally, fix an $N\in\mathbb{N}$ such that $1/N<\varepsilon$ and cut each $J_j(a(p+n))$ to N equally long intervals $K_j(1),\dots,K_j(N)$ and put $\mathcal{E}=\bigcup_j\{K_j(1),\dots,K_j(N)\}\cup\mathcal{M}$. Since any code from $\operatorname{Code}(2^n,\mathcal{D})$ contains at most $2^{a(p)}$ symbols from $\mathcal{J}$ we have

$$\operatorname{Card}\operatorname{Code}(2^n,\mathcal{E})\leq\operatorname{Card}\operatorname{Code}(2^n,\mathcal{D})\cdot N^{2^{a(p)}}=O\left(2^{n^2}\right).\tag{7}$$

It is obvious that $\operatorname{Span}(T,f|\cup\operatorname{Orb}_f J(a(p)),\varepsilon,2^n)\leq\operatorname{Card}\operatorname{Code}(2^n,\mathcal{E})$. In fact, if for each code from $\operatorname{Code}(2^n,\mathcal{E})$ we choose one point having this code then the set of all chosen points is an $(T,f|\cup\operatorname{Orb}_f J(a(p)),\varepsilon,2^n)$-span. □

Theorem 6. *Let T be an arbitrary increasing sequence of nonnegative integers. Then there is a chaotic map $f\in C(I,I)$ such that $h_T(f)=0$.*

Proof. For T define f as in the beginning of this section. Fix $\varepsilon>0$. We will show that $\limsup_{n\to\infty}\frac{1}{n}\log\operatorname{Span}(T,f,\varepsilon,n)=0$.

Let $J(a(p))$ be the interval from Lemma 5. To bound above $\operatorname{Span}(T,f,\varepsilon,2^n)$ we use Lemma 5 but we have to take into account also T-trajectories starting in intervals contiguous to $\operatorname{Orb}J(a(p))$, i.e. in intervals from the family $\mathcal{C}=\{C(1),C_1(2),C_2(2),\dots,C_1(a(p)),\dots,C_{2^{a(p)-1}}(a(p))\}$. Here $C(1)$ is contiguous to $\operatorname{Orb}_f J(1)$, then intervals $C_1(2),C_2(2)$ are remaining two contiguous intervals to $\operatorname{Orb}_f J(2)$, etc. Evidently, $i>j$ implies $f^kC_r(i)\cap C_s(j)=\emptyset$ for any $k\in\mathbb{N}$, $1\leq r\leq 2^{j-1}$ and $1\leq s\leq 2^{i-1}$. A code from $\operatorname{Code}(m,\mathcal{C})$ is a finite sequence of the form

$$\begin{aligned}&\big(C_{r(i_1,1)}(i_1),\dots,C_{r(i_1,m_1)}(i_1),C_{r(i_2,1)}(i_2),\dots,C_{r(i_2,m_2)}(i_2),\dots\\&\qquad\dots,C_{r(i_k,1)}(i_k),\dots,C_{r(i_k,m_k)}(i_k)\big)\end{aligned}\tag{8}$$

where $m_1+\dots+m_k=m$ and $i_1<\ldots<i_k$. We are going to estimate $\operatorname{Card}\operatorname{Code}(m,\mathcal{C})$.

Obviously, $k\leq m$ and $i_1<\cdots<i_k\leq a(p)$. Hence $k\leq\min\{m,a(p)\}$. Fix $k,m_1,\dots,m_k,i_1,\dots,i_k$. First, realize that each subsequence $(C_{r(i_j,1)}(i_j),\dots,C_{r(i_j,m_j)}(i_j))$ is uniquely determined by $C_{r(i_j,1)}(i_j)$. This follows from the fact that for any $l\in\mathbb{N}$, $1\leq i\leq a(p)$, $1\leq r\leq 2^{i-1}$ there is exactly one s, $1\leq s\leq 2^{i-1}$,

such that $f^l C_r(i) \cap C_s(i) \neq \emptyset$. Since there are 2^{i_j-1} possibilities for the choice of $C_{r(i_j,1)}(i_j)$, for fixed $k, m_1, \dots, m_k, i_1, \dots, i_k$ we obtain at most $2^{i_1-1} \cdot \dots \cdot 2^{i_k-1} = 2^{i_1+\dots+i_k-k} \leq 2^{a(p)^2}$ sequences of the form (8).

There are $\binom{m-1}{k-1}$ ways how to write m in the form $m_1 + \dots + m_k$ and $\binom{a(p)}{k}$ ways of chosing k positive integers $i_1 < \dots < i_k \leq a(p)$. From all this we obtain that

$$\begin{aligned} \operatorname{Card}\operatorname{Code}(m, \mathcal{C}) &\leq \sum_{k=1}^{\min\{m,a(p)\}} \binom{m-1}{k-1} \cdot \binom{a(p)}{k} \cdot 2^{a(p)^2} \\ &\leq 2^n \cdot 2^{n\cdot(a(p)-1)} \cdot a(p)! \cdot 2^{a(p)^2} = O\left(2^{a(p)\cdot n}\right). \end{aligned} \tag{9}$$

Cut each interval from the family $\mathcal{C}$ into N equally long intervals and denote by $\mathcal{C}_{cut}$ the family of all intervals obtained in such a way. We are going to estimate $\operatorname{Card}\operatorname{Code}(m, \mathcal{C}_{cut})$. Take any $j \in \{1, \dots, k\}$ and consider all points having the code $\left(C_{r(i_j,1)}(i_j), \dots, C_{r(i_j,m_j)}(i_j)\right)$, a subcode of (8). The number of different codes from $\operatorname{Code}(m_j, \mathcal{C}_{cut})$ belonging to the considered points is at most $N + (m_j - 1)\cdot(N-1) \leq m_j \cdot N$. This follows from the fact that f is strictly monotone on each $C_r(i)$ which implies that $f^{-l}C_r(i) \cap C_s(i)$ is either an interval or the empty set and in the former case f^l is strictly monotone on this interval. Therefore the number of different codes from $\operatorname{Code}(m, \mathcal{C}_{cut})$ belonging to all points whose code from $\operatorname{Code}(m, \mathcal{C})$ is the sequence (8) is at most $m_1 \cdot N \cdot \dots \cdot m_k \cdot N \leq 2^{a(p)\cdot n} \cdot N^{a(p)}$. Since this upper bound does not depend on the integers $k, m_1, \dots, m_k, i_1, \dots, i_k$, from (8) we have, by (9),

$$\operatorname{Card}\operatorname{Code}(m, \mathcal{C}_{cut}) \leq 2^{a(p)\cdot n} \cdot N^{a(p)} \cdot \operatorname{Card}\operatorname{Code}(m, \mathcal{C}) = O\left(2^{2\cdot a(p)\cdot n}\right). \tag{10}$$

Now we can bound above the number of all codes from $\operatorname{Code}(2^n, \mathcal{E} \cup \mathcal{C}_{cut})$. Each such a code consists of a (possibly empty) block of symbols from $\mathcal{C}_{cut}$ followed by a (possibly empty) block of symbols from $\mathcal{E}$. Therefore

$$\operatorname{Card}\operatorname{Code}(2^n, \mathcal{E} \cup \mathcal{C}_{cut}) \leq \sum_{m=0}^{2^n} \operatorname{Card}\operatorname{Code}(m, \mathcal{C}_{cut}) \cdot \operatorname{Card}\operatorname{Code}(2^n - m, \mathcal{E}).$$

By (7) and (10),

$$\begin{aligned} \operatorname{Card}\operatorname{Code}(2^n, \mathcal{E} \cup \mathcal{C}_{cut}) &= (2^n + 1) \cdot O\left(2^{2\cdot a(p)\cdot n}\right) \cdot O\left(2^{n^2}\right) \\ &= O\left(2^{n^2 + (2\cdot a(p)+1)\cdot n}\right). \end{aligned}$$

Then, since $\operatorname{Span}(T, f, \varepsilon, 2^n) \leq \operatorname{Card}\operatorname{Code}(2^n, \mathcal{E} \cup \mathcal{C}_{cut})$

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{Span}(T, f, \varepsilon, n) &\leq \limsup_{n\to\infty} \frac{1}{2^{n-1}} \log \operatorname{Span}(T, f, \varepsilon, 2^n) \\ &\leq \limsup_{n\to\infty} \frac{1}{2^{n-1}} \log O\left(2^{n^2+(2\cdot a(p)+1)\cdot n}\right) = 0. \end{aligned}$$

□

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